MONTHLY NOTICES

OF THE

ROYAL ASTRONOMICAL SOCIETY

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MEETING OF 1936 JANUARY 10

Mr. J. H. Reynolds, President, in the Chair.

The election by the Council of the following Fellows was duly confirmed :—

Clabon Walter Allen, M.Sc., Solar Physics Observatory, Cambridge ;
Francis Parry Beech, National Provincial Bank, Bethesda, Bangor ;
Captain Guy Bomford, R.E., Survey of India, Dehra Dun, India ;
Percy Byerly, Professor of Seismology, University of California, Berkeley, Cal., U.S.A. ;
Léon Cap, 151 Post Box, Antwerp, Belgium ;
Rev. Ronald William Cartmel, B.A., 28 Birchwood Road, Wolverhampton ;
John Edward Daly, 5 Payne Street, Barnsbury, London, N. 1 ;
Dharmasiri Jayampathie Jayasinghe, B.Sc., Colombo Observatory, Ceylon ;
Alvapillai Ponnusamy Kandasamy, B.Sc., Colombo Observatory, Ceylon ;
Peter Kennedy, M.I.E.E., A.Amer.I.E.E., 77 Edmund Avenue, Unley, S. Australia ;
Zdenek Kopal, 1423 U. Pernikarky, Prague ;
Miss I. Lehmann, Chief Seismologist, Geodetic Institute, Copenhagen ;
Major Earl Pitt, B.A., B.Sc., B.Litt., Meadway, Whitchurch, Herefordshire ; and
Kiyofusa Sotome, Astronomical Observatory, Tokyo.

The following candidates were proposed for election as Fellows of the Society, the names of the proposers from personal knowledge being appended :—

Max Krook, Gonville and Caius College, Cambridge (proposed by F. J. M. Stratton); and

Arthur Mulligan, M.Sc., 6 Keswick Road, St. Helens, Lancs (proposed by A. Fletcher).

Seventy presents were announced as having been received since the last meeting, including :—

Dearborn Observatory, Annals, vol. 3.—Stellar Parallaxes.

Lick Observatory Publications, vol. 19.—Research Surveys of the Orbits and Perturbations of Minor Planets 1 to 1091 from 1801·0 to 1929·5, by A. O. Leuschner.

The President announced that the Gold Medal of the Society had been awarded to Professor H. Kimura, Director of the Latitude Station at Mizusawa, Japan, for his valuable work on the Variation of Latitude.

THE RED SHIFT OF THE IRON LINES AT THE EDGE OF THE SUN.

J. Evershed, F.R.S.

In his paper on the gravitational displacement of the solar lines published in the *Astrophysical Journal*, **67**, 1928, the late Dr. St. John states that "lines of iron to the number of 133 give at the limb a mean red displacement for high- and low-level lines which is 0·0015 A. greater than that calculated from the theory of general relativity." This small residual he considered to be a true limb effect if real, but apparently of no great significance with regard to his main conclusion that "the causes of the differences at the centre of the Sun between solar and terrestrial wave-lengths are the slowing up of the atomic clock in the Sun according to Einstein's theory of general relativity, and radial velocities of moderate magnitude and in probable directions or equivalent conditions, whose effects vanish at the edge of the Sun."

My own measures of solar and arc lines in general sunlight and at the centre of the Sun's disc are in good agreement with those of St. John; and the conclusion I arrived at in 1923 was that "there is very little doubt that the Einstein effect is present in the solar spectrum." This was published in the *Observatory*, **66**, 304, 1923, in which article I drew attention to the considerable excess of shift shown by certain high-level lines in the ultra-violet at the Sun's limb.

My very numerous measures of the strong and weak iron lines in the *H* and *K* region give a very much larger mean residual shift than is given by St. John, and they also show that the high- and low-level lines which at the centre of the Sun give very appreciable plus and minus residuals respectively give nearly equal shifts near the limb. There are two reasons why the shifts of the weak low-level lines are probably under-estimated. In the first place, my measures of the relative shift of these lines at numerous points across the

disc show that the increase of wave-length from the centre towards the limb becomes very rapid as the limb is approached, whereas the strong high-level lines show only a very small increase. As the measures are made always at a little distance within the limb, an appreciable correction to the limb is necessary for the weak lines. A further unknown correction due to scattered skylight, mainly from the centre of the Sun, is necessary, and this is especially the case for weak lines in the ultra-violet.

It may be for these reasons that St. John obtained very small residuals in the ultra-violet for the weaker lines. It is in any case quite certain that the group of *Fe* lines that I have measured in the *H* and *K* region give a true limb effect of at least +0·006 A., and this includes both high-level and low-level lines. It is also noteworthy that I have found the same excess over the Einstein effect in the calcium lines H_3 and K_3, not only as absorption lines in the chromosphere but also as emission lines high up in the prominences, where there can be no possible effect on the wave-length which might be thought to occur during the passage of the light through the gases of the Sun's atmosphere. This "limb effect," whatever its origin, is thus found well outside of the limb.

The discrepancy in our results has led me recently to make new measures of limb and arc in the red region, where atmospheric scattering is less than in the more refrangible part of the spectrum. In the year 1921 I measured the limb – arc shifts of five *Fe* lines, not subject to pole effect, in 17 grating spectra obtained at Kodaikanal. These gave such large values of the shift that I had doubts of its reality, but now I have obtained some excellent spectra at Ewhurst, and these give similar results. The difficulty in this region is the necessity for very long exposures in order to get good images of the arc lines. The iron arc requires about 30 times the exposure needed on the Sun's limb, although at the other end of the visible spectrum it is the Sun that requires the longer exposure. This difficulty almost precludes the use of the liquid prism : I have therefore employed a solid 60° prism of $2\frac{1}{2}$-inch aperture backed by a 30° prism, and to get sufficient dispersion the light is made to traverse the two prisms six times to and fro, the resolving power being equivalent to nine 60° prisms of that aperture. With this equipment the exposure on the Sun's edge in winter is 15 seconds, and on the *Fe* arc about 8 minutes, the linear dispersion being 1·3 mm. = 1 A. This is ample for the purpose, and the spectra are well suited to the extremely accurate method of measuring by superposing positive images reversed on the negatives. I give in Table I the results of these measures.

TABLE I

λ	17 Grating Spectra $\Delta\lambda$	8 Prism Spectra $\Delta\lambda$	Weighted Mean $\Delta\lambda$
6136·631	+·026	+·027	+·026
6137·709	+·024	+·027	+·025
6191·577	+·025	+·030	+·027
6230·742	+·022	+·029	+·024
6252·572	+·022	+·027	+·024

These are the shifts observed at points on the Sun about ·98 of a radius from the centre, and are therefore under-estimates. The correction, however, is not likely to exceed ·001 A. The shifts have been corrected for the pressure in the arc assuming the Sun to be at zero pressure. This gives a true limb effect of at least +0·012 A.

We may here consider also the limb effect of the hydrogen line *Hα*, which, notwithstanding its width of about 1 A. in the Sun, can be measured more accurately than the other hydrogen lines. The difference of wave-length of this line at the centre of the Sun compared with the vacuum tube is +0·023 A.,* and to this should be added the limb-centre shift of +0·002 A. measured by Adams.† The limb – vacuum wave-length is therefore

	+0·025 A.
Subtract Einstein effect	+0·014 ,,
Limb Effect	+0·011 A.

Thus the limb effect is the same order of size in this case also.

In measuring the shifts at the Sun's edge St. John made rather a curious selection of lines. For those of solar intensity 8–25 only the ultra-violet region between λλ 3787 and 3906 was used, and for lower intensities the same ultra-violet region, but in addition lines between λλ 5000 and 5567. He left out of account the entire range of the blue and violet and the red. Although he did not measure the limb shifts in these regions he gives long lists at the centre of the Sun, including almost the whole length of the visible spectrum. These are comparisons of the normal solar spectrum and the vacuum arc, and it so happens that for over a hundred of these lines we can deduce the limb shifts by adding the limb – centre shifts previously determined at Mt. Wilson and at Kodaikanal.‡ In Table II, which follows, I give the results for 105 *Fe* lines between λλ 3748 and 6495 that correspond in the various tables. These are all of St. John's classes A and B, but had I included his classes C and D, which are less trustworthy owing to pole effect, 30 lines would have been added to the list ; and although these lines give a less uniform and slightly smaller limb shifts, the mean excess given by them over relativity is +0·003 A. No appreciable difference would therefore have been made in the general result.

In all this list only 18 lines occur in St. John's direct measures of limb shift. Four of these are in the ultra-violet, and 14 are in the green. The agreement with this list is very good in both regions, and only 4 lines in the green show a marked deficiency in the direct measures.

The figures in the last column of the table are also in good agreement with my values of the limb shift in the two spectral regions that I have specially studied. In the red region also the mean values from the 25 spectra

* This is the difference between Curtis's value of the wave-length of *Hα* (*Proceedings of the Royal Society*, A, **90**, 1914) and the solar wave-length in the Revision of Roland.

† *Astrophysical Journal*, **31**, 45, 1910. This is the observed shift between centre and limb. Adams applied corrections to his observed values which are not justified by present knowledge of the very low pressure in the chromosphere.

‡ *Astrophysical Journal*, **31**, 37–45, and *Kodaikanal Bulletin*, **39**, 72–4, 1911.

TABLE II

λ Revised Roland	Int.	Limb − ⊙ Mt. Wilson	Limb − ⊙ Kodaikanal	⊙ − Vacuum Arc St. John	Limb − Vacuum Arc A/1000
3748·273	10	+ 2	...	+10	+12
3834·235	10	4	...	11	15
3846·419	2	5	...	10	15
3871·760	2	8	...	10	18
3895·669	7	...	− 3	10	7
3902·958	10	...	− 2	11	9
3906·492	10	...	+ 4	9	13
3920·271	10	...	6	11	17
3922·925	12	...	7	12	19
3925·653	5	...	7	7	14
3927·935	8	...	2	13	15
3930·310	8	...	2	12	14
3931·131	1	...	8	7	15
3935·828	2	...	4	13	17
3937·339	3	...	3	9	12
3948·787	4	...	8	9	17
3949·963	5	6	6	7	13
3956·688	6	7	6	9	15
3966·075	3	...	4	9	13
3969·270	10	...	4	11	15
3977·752	6	6	7	9	16
3986·182	3	...	6	6	12
3998·060	4	...	7	4	11
4009·719	3	...	9	4	13
4021·872	5	9	14	2	14
4045·827	30	...	8	13	21
4062·451	5	...	8	6	14
4063·607	20	8	8	11	19
4071·751	15	...	3	11	14
4076·639	4	...	7	3	10
4118·557	5	...	6	8	14
4127·615	4	...	5	3	8
4134·687	5	...	6	5	11
4143·880	15	8	8	9	17
4154·507	4	...	6	4	10
4175·645	5	...	6	5	11
4181·766	5	...	7	8	15
4202·042	8	5	3	11	15
4220·349	3	8	7	5	12
4258·326	2	6	...	7	13
4271·776	15	6	...	12	18
4282·413	5	7	8	8	15
4291·475	2	7	...	8	15
4307·914	6	...	10	8	18
4325·777	8	3	− 2	13	14
4337·057	5	8	+ 8	8	16
4352·745	4	...	6	8	14
4369·781	4	...	12	8	20
4375·946	6	5	4	14	18
4383·559	15	...	1	11	12
4404·763	10	...	− 1	10	9
4415·137	8	...	+ 3	11	14

λ Revised Roland	Int.	Limb – ⊙ Mt. Wilson	Limb – ⊙ Kodaikanal	⊙ – Vacuum Arc St. John	Limb – Vacuum Arc A/1000
4427·319	5	...	10	7	17
4430·624	3	8	8	7	15
4442·351	6	8	4	9	15
4447·730	6	5	4	10	15
4454·390	3	7	7	7	14
4461·662	4	7	7	8	15
4466·564	5	...	8	11	19
4489·750	4	7	...	8	15
4494·575	6	10	6	7	15
4522·640	3	10	...	8	18
4531·160	5	7	8	8	15
4547·856	3	8	4	6	12
4549·476	2	8	...	6	14
4592·661	4	...	6	6	12
4602·951	6	...	4	7	11
4619·299	3	...	4	6	10
4638·019	4	...	5	4	9
4678·857	6	...	6	5	11
4733·599	4	10	...	5	15
4741·538	3	8	...	7	15
4745·809	4	9	...	6	15
4786·816	2	8	...	6	14
4789·660	3	7	...	7	14
4924·779	3	8	...	6	14
4994·139	3	8	...	6	14
5083·347	4	9	...	5	14
5171·612	6	...	– 2	13	11
5194·951	4	8	+ 4	8	14
5198·718	3	7	...	8	15
5216·285	3	...	3	9	12
5328·544	4	...	7	12	19
5332·910	4	10	...	8	18
5405·787	6	9	...	10	19
5429·708	6	9	...	9	18
5434·536	5	10	...	12	22
5446·926	6	10	...	9	19
5455·626	4	8	...	14	22
5497·528	5	12	...	9	21
5501·479	5	10	...	12	22
5506·793	5	11	...	12	23
5956·709	4	10	...	11	21
6065·499	7	12	...	11	23
6136·631	8	13	...	13	26
6173·348	5	14	...	10	24
6191·577	9	14	...	13	27
6213·443	6	14	...	9	23
6230·742	8	11	...	12	23
6252·572	7	13	...	10	23
6265·148	5	13	...	8	21
6318·036	6	14	...	12	26
6380·756	4	8	...	8	16
6430·863	5	12	...	9	21
6495·001	8	13	...	13	26

given in Table I agree almost exactly with the same lines in Table II. These comparisons are given in detail in Table III, where my values in column 3, it must be understood, have been obtained with an open arc at ¾ atmosphere at Kodaikanal and 1 atmosphere at Ewhurst. A correction has therefore been applied to allow for the atmospheric pressure in the arc, assuming, of course, a zero pressure in the Sun. These corrections are derived from the pressure shift measures of Adams and Gale,* and Babcock.†

TABLE III

Group 1

Wave-length	Limb – Vacuum Arc from Table II A/1000	Limb – Vacuum Arc Evershed A/1000	No. of Spectra Measured
3895·669	7	14	32
3902·958	9	13	38
3906·492	13	16	43
3920·271	17	15	78
3922·925	19	15	78
3925·653	14	13	21
3927·935	15	15	76
3930·310	14	16	72
3935·828	17	17	28
3949·963	13	14	44
3956·688	15	14	56
3966·075	13	13	49
3969·270	15	16	59
3977·752	16	13	39
Means 3932	+0·0141 A.	+0·0146 A.	
Subtract Einstein effect	−0·0083 ,,	−0·0083 ,,	
Limb effect	+0·0058 A.	+0·0063 A.	

Group 2

Wave-length	Limb – Vacuum Arc from Table II A/1000	Limb – Vacuum Arc Evershed A/1000	No. of Spectra Measured
4375·946	18	15	35
4383·559	12	11	19
4404·763	9	12	20
4415·137	14	10	20
4427·319	17	12	29
4442·351	15	17	28
4447·730	15	20	28
4454·390	14	13	25
4461·662	15	13	28
4466·564	19	18	28
4489·750	15	12	18
Means 4433	+0·0148 A.	+0·0139 A.	
Subtract Einstein effect	−0·0094 ,,	−0·0094 ,,	
Limb effect	+0·0054 A.	+0·0045 A.	

* *Astrophysical Journal*, **35**, 10, 1912. † *Ibid.*, **67**, 240, 1928.

Group 3

Wave-length	Limb – Vacuum Arc from Table II A/1000	Limb – Vacuum Arc Evershed A/1000	No. of Spectra Measured
6136·631	26	26	25
6137·709	...	25	25
6191·577	27	27	25
6230·742	23	24	25
6252·572	23	24	25
Means 6203 omitting 6137	+0·0248 A.	+0·0252 A.	
Subtract Einstein effect	−0·0131 „	−0·0131 „	
Limb effect	+0·0117 A.	+0·0121 A.	

While these comparisons show a general agreement between the combined measures in Table II and direct measures of the limb shift, considerable errors may occur where but few plates have been measured of solar lines. There are always local currents, either parallel to or radial to the Sun's surface which upset our measures, and only mean values from at least 20 spectra can give reliable results. It is possible that future work may indicate considerable corrections for some of the lines. The irregular way in which limb shift and limb effect vary with wave-length seems to suggest this. In a general way from the violet to the red the shift increases as a Doppler or Einstein effect increases, but in the blue between $\lambda\lambda$ 4500 and 4900 the values are much too small. This is shown in Table IV, where I give the mean shift and mean limb effect for each consecutive set of ten lines, excepting the last set containing 15 lines. The first column gives the corresponding mean wave-length. It will be noted that in the red the limb effect is nearly equal to the Einstein effect; in other words, the shift at the limb is probably equal to the Einstein effect × 2.

TABLE IV

Mean Wave-length	Mean Limb Shift A/10,000	Mean Limb Effect A/10,000
3877	139	57
3941	147	64
4028	143	58
4172	136	48
4330	160	68
4444	148	54
4578	127	30
4917	140	36
5375	185	71
6121	230	106

It is to be hoped that arc and Sun limb spectra may be obtained at future solar eclipses, either total or annular, in order to get values of the limb effect freer from the effects of scattered light than is possible under ordinary conditions. Apart from the fact that the values given in these tables are

probably under-estimated, it is certain that the limb effect is very far from being a negligible quantity. Unfortunately it seems to land us on the horns of a dilemma, for we must either suppose that the excess over relativity is a Doppler effect, in which case we have to imagine the solar gases to be receding at all times from the Earth, or that the general theory of relativity requires a factor of 2 to be introduced in the slowing down of the atomic clocks. I believe that no astronomer will care to impale himself on either of these horns, but in that case he has to find an explanation of the limb effect which applies equally to the absorption lines of the reversing layer and to the emission lines in the prominences beyond the limb.

Ewhurst, Guildford,
Surrey :
1936 *January* 1.

ON THE EFFECT OF ROTATING THE SECONDARY MIRROR OF A CŒLOSTAT.

T. W. Wormell, M.A., Ph.D.

(Communicated by The Director, Solar Physics Observatory, Cambridge)

A cœlostat consists of a plane mirror rotating about an axis, in its own plane, and parallel to the Earth's axis of rotation, with an angular velocity equal to half the diurnal motion of the celestial object (Sun or star) which is being observed. For the sake of brevity attention is confined to the Sun; the mirror then rotates once in forty-eight hours. The beam from any point on the Sun is reflected in a constant direction, *i.e.* the field does not rotate. By turning the mirror about its axis of rotation, the reflected beam coming originally from a point on the Sun's disc, of declination δ, may be directed to any point on the parallel of declination $-\delta$, but in no other direction.

In order that the reflected beam may be fed into an immovable telescope at any period of the year and, in general, for convenience, it is necessary to introduce a stationary secondary plane mirror. The centre of this mirror must be on the axis of the telescope. The relative position of the two mirrors must be capable of adjustment in order that, for all possible values of the Sun's declination, the beam from the cœlostat may be directed on to the secondary mirror. This may be effected either by moving the secondary mirror along the axis of the telescope, or by moving the cœlostat itself. To avoid shadows cast by parts of the apparatus, or by the protective shelter, the primary mirror is also commonly provided with a horizontal east and west motion.

The final adjustment of the direction of the beam in the telescope is conveniently made by small rotations of the secondary mirror about two perpendicular axes. Such adjustments can readily be controlled from a distance.

Moreover, in some recent work at the Solar Physics Observatory, Cambridge, it was found convenient to drift the solar image slowly at a known rate, by rotating the secondary mirror by means of a small synchronous electric motor and suitable gearing. The main purpose of this note is to examine the nature of the motion of the image in the focal plane of the telescope when the secondary mirror is rotated with a definite angular

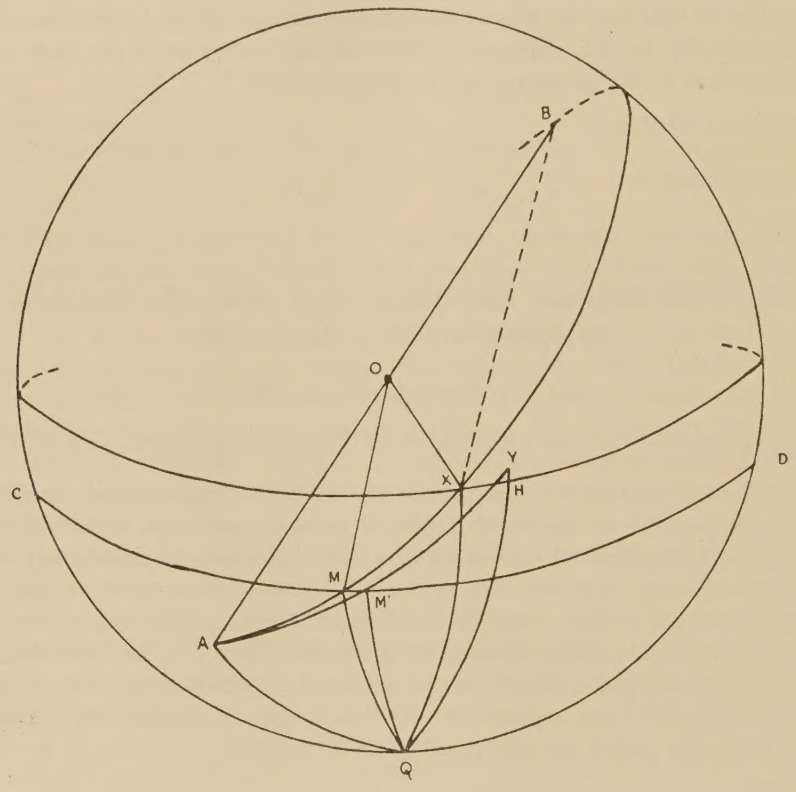

Fig. 1.

velocity. I have been unable to discover any treatment of this point in the literature; the problem bears certain resemblances to the geometry of the siderostat as considered, for example, by Plummer.*

For simplicity one particular case is considered. It is assumed that the beam after reflexion travels horizontally and due north. The analysis may readily be extended to other cases, and in particular to the case of a vertical telescope, by a suitable change in the meaning of the symbols employed. We consider the intersections of the various directions concerned with the celestial sphere. In fig. 1, *AO* represents the stationary direction of the beam from the centre of the Sun's disc after reflexion at the cœlostat, *O* the centre of the secondary mirror at the centre of the sphere, *OX* the beam reflected from the secondary mirror, *OM* the normal

* H. C. Plummer, *M.N.*, **61**, 459, 1901.

to the secondary mirror. M bisects the great circle arc AMX. We consider the motion of the reflected beam OX in two cases. First, as the mirror at O rotates about a vertical axis with constant angular velocity, *i.e.* as M describes uniformly the horizontal small circle CMD. Secondly, the mirror at O may be rotated about a horizontal axis in its own plane, *i.e.* M travels along the vertical circle QM.

If AO be produced to meet the sphere again in B, it is readily seen that BX is parallel to OM; thus if M travels round CD, BX generates a right circular cone, and the path of X on the surface of the sphere is the intersection of the sphere with a right circular cone whose vertex is at B, axis vertical and semi-vertical angle equal to the inclination of OM to the vertical. Secondly, if M travels round the vertical circle QM, BX rotates around B in a plane parallel to the plane of the great circle QM, and X traces on the sphere a small circle, whose plane is vertical.

We are, however, mainly interested in the behaviour of X when OX is approximately horizontal and directed towards the north. We proceed to discuss this analytically. Hartmann * in a thorough discussion of the geometry of the siderostat gives a general expression for the change in the direction of the reflected beam from a plane mirror due to small changes in the direction of the normal and of the other variables. The particular case we require may be obtained in a more convenient form, however, and much more simply. We express all positions on the sphere in terms of the position of the normal OM when this has been so adjusted as to reflect the stationary beam AO to the north along the horizontal line OX. We define the position of M by the great circle arc from the nadir point Q, $QM = z$, and the azimuth $\alpha = \widehat{MQX}$.

Solving in succession the spherical triangles MQX and AQM, it is readily found that

$$\cos AQ = \sin 2z \cos \alpha$$

and

$$\sin \widehat{AQX} = \frac{\sin 2\alpha \sin^2 z}{\sqrt{1 - \sin^2 2z \cos^2 \alpha}},$$

defining the constant direction of the beam AO from the cœlostat in terms of z and α.

We now consider the effect of a small change in the position of M. First, let the secondary mirror be rotated about a vertical axis so that the normal moves from OM to OM', where the co-ordinates of M' are $(z, \alpha - \Delta\alpha)$, the position of A remaining fixed. The reflected beam moves from OX to OY, where Y is on the great circle arc $AM'Y$ and $AM' = M'Y$. Let the vertical circle QY intersect the horizontal plane through O in H.

Solving successively the spherical triangles AQM' and $M'QY$, we find that, to the first order in $\Delta\alpha$,

$$\cos QY = -\Delta\alpha \sin 2z \sin \alpha,$$

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* W. Hartmann, *Ast. Abh. Hamburg Bergedorf*, iv, 1, p. 9, 1928.

whence, since

$$QH = \frac{\pi}{2},$$

$$HY = \Delta\alpha \sin 2z \sin \alpha. \qquad (1)$$

Further,

$$\sin \widehat{M'QY} = \sin \alpha - \Delta\alpha \cos \alpha \cos 2z,$$

whence, since

$$\widehat{M'QX} = \alpha - \Delta\alpha,$$

$$\widehat{XQY} = XH = \Delta\alpha \, 2 \sin^2 z. \qquad (2)$$

We have thus obtained the horizontal and vertical components of the displacement of the reflected beam, which was initially horizontal and directed due north, on the celestial sphere or on a vertical tangent plane to the sphere at X, due to the small change $\Delta\alpha$ in the azimuth of the secondary mirror. It follows that the resultant displacement of the image, measured in radians about O, is equal to

$$\Delta\alpha \, . \, 2 \sin z \sqrt{(\sin^2 z + \cos^2 z \sin^2 \alpha)},$$

and that the direction of motion of the image is inclined at an angle $\tan^{-1}(\cot z \, . \sin \alpha)$ to the horizontal. The sense of the angle is clear from fig. 1 ; if the secondary mirror at O is west of the cœlostat and is rotated about OQ in such a sense as to deflect the beam OX towards the west, the beam rises above the horizontal ; if the mirror be rotated in the opposite sense the beam sinks below the horizontal.

Secondly, we consider the effect of rotating the secondary mirror about a horizontal axis in its own plane, *i.e.* α remains constant while z varies. In this case M moves along the vertical circle QM. It is readily found that the corresponding motion of X is always vertical (as is obvious geometrically) and that the displacement of the reflected beam is $\Delta z \, . \, 2 \cos \alpha$.

To complete the account of the behaviour of the image we proceed to consider its orientation. Unless the cœlostat and secondary mirror are in the same meridian, a north and south line in the image will not be vertical. In fig. 2 the letters have the same meanings as in fig. 1, with the addition that P is the North Pole of the celestial sphere. The beam from the centre point of the Sun's disc after its first reflexion travels along AO ; the image of P in the primary mirror coincides with P. The beam AO after reflexion at the secondary mirror is reflected horizontally along OX, while the beam PO is reflected to P' on the great circle arc PMP', where $PM = MP'$.

Drawing the great circle arc XP', the angle $P'XQ$ is the inclination to the vertical of a north and south line in the image.

PX is the latitude ϕ ; let PM, the north polar distance of M, be ξ. We have

$$\cos \xi = \sin z \cos \alpha \cos \phi - \cos z \sin \phi$$

and

$$\sin \widehat{MPX} = \frac{\sin z \sin \alpha}{\sin \xi}.$$

Then, considering the spherical triangle $PP'X$, we have by a well-known formula

$$\cot \widehat{PXP'} = \frac{\sin \phi \sin \xi \cot 2\xi - \cos \phi \sqrt{(\sin^2 \xi - \sin^2 z \sin^2 \alpha)}}{\sin z \sin \alpha}. \qquad (3)$$

Two particular cases are of some interest. If the beam AO from the cœlostat is in a horizontal plane, since OX is by definition horizontal, the normal OM is also in a horizontal plane, *i.e.* z is $\pi/2$. Thus if the secondary

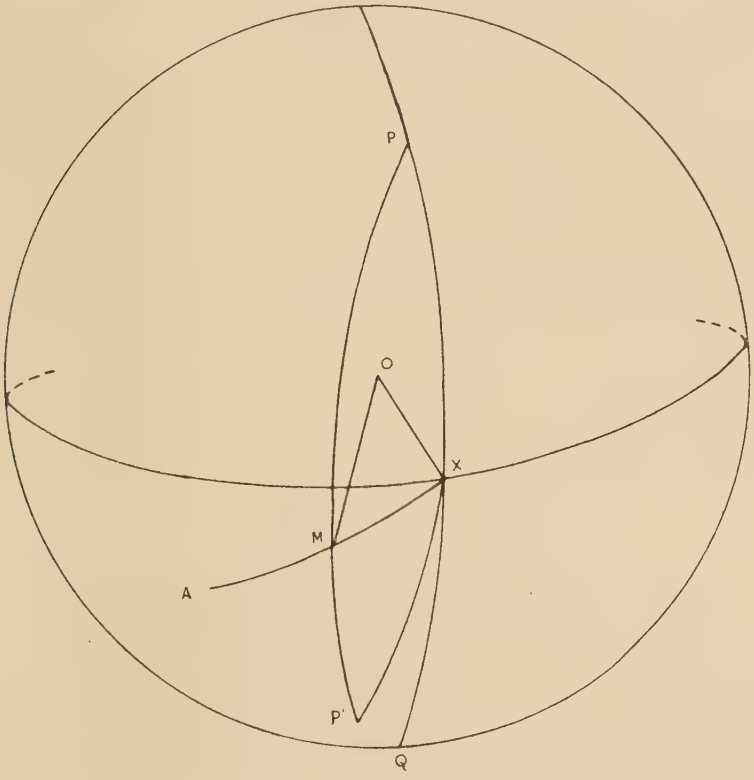

Fig. 2.

mirror be rotated about a vertical axis, the motion of the image is horizontal and equal to $2\Delta\alpha$, (1) and (2). This is, of course, simply an example of a common physical method of measuring small changes of angle as in a reflecting galvanometer.

Again, if the cœlostat and secondary mirror are both in the same meridian, $\alpha = 0$. A rotation of the secondary mirror about a vertical axis again causes a horizontal motion of the image ; the amount of the motion is in this case $\Delta\alpha \,.\, 2 \sin^2 z$, (1) and (2). Further, in this case a north and south line in the image is vertical, (3).

For a cœlostat at a given place, the value of z and also of α (unless the instrument is used in the position $\alpha = 0$) will vary with the Sun's declination. Thus both the direction and magnitude of the velocity of the image,

for a given angular velocity of the secondary mirror about a vertical axis, will vary from day to day. The orientation of the image will also vary. If the secondary mirror be provided with divided circles attached to its axes of rotation, the values of z and a may be read off directly, and the nature of the motion and orientation of the image deduced. In any case the direction of the normal to the secondary mirror may be calculated by simple trigonometry from the relative position of the two mirrors.

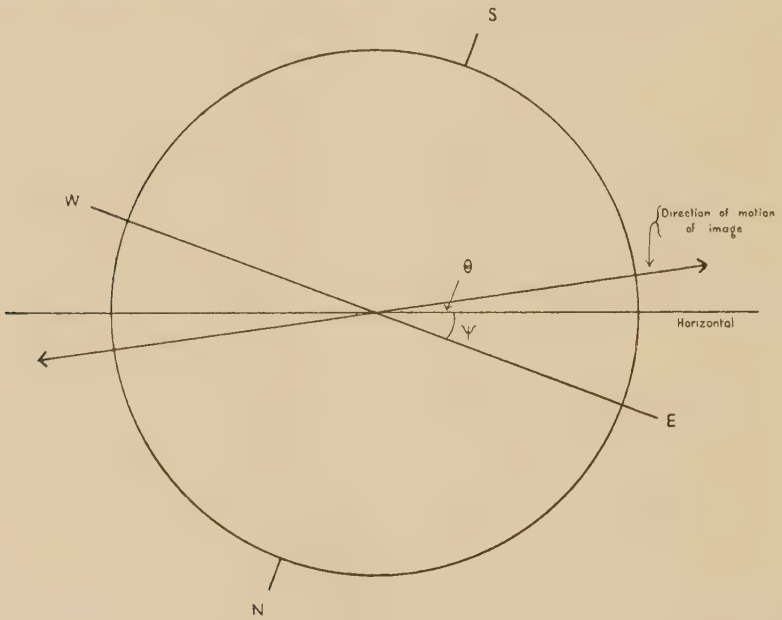

FIG. 3.—*Orientation and direction of motion of image in focal plane as viewed by observer looking horizontally southwards, the cœlostat being to the east of the axis of the telescope.*

For purposes of illustration some figures are tabulated for the McClean cœlostat of the Solar Physics Observatory, latitude 52° 13′. In this instrument * the centre of the secondary mirror remains always at the same point on the axis of the horizontal telescope. The cœlostat is mounted on horizontal rails, giving an east and west motion of 50 cm. on either side of the axis of the telescope. It may also be moved up and down on a slide parallel to the polar axis. The table contains the following quantities for this instrument with the cœlostat at the eastern extremity of its horizontal motion, for the two solstices and for the equinox :—

	Co-ordinates of Normal to Secondary Mirror		Ratio of Velocity of Image to Velocity of Mirror	Inclination of Motion to Horizontal	Inclination of East-West Line in Image to Horizontal	Inclination of Motion to East-West Line in Image
	z	a		θ	ψ	$\theta+\psi$
	° ′	° ′		° ′	° ′	° ′
Summer Solstice	84 33	12 47	$1{\cdot}98_2$	1 13	14 26	15 39
Equinox	72 53	15 20	$1{\cdot}83_2$	4 39	9 39	14 18
Winter Solstice	60 56	17 0	$1{\cdot}54_8$	9 14	4 6	13 20

* J. B. Hubrecht, *Annals of the Solar Physics Observatory*, **3**, 1, 1915.

Columns 1 and 2 contain the co-ordinates z and a of the normal to the secondary mirror, when correctly adjusted to send the beam from the centre of the Sun's disc parallel to the axis of the horizontal telescope. Column 3 gives the ratio of the angular velocity of the central point of the image about the secondary mirror to the angular velocity of that mirror about a vertical axis. Column 4 gives the inclination θ of the motion to the horizontal, column 5 the inclination ψ of an east and west line in the image to the horizontal, and column 6 the inclination $(\theta+\psi)$ of the direction of motion to the east and west direction in the image.

The sense of the angles is illustrated in fig. 3, which represents the image in the focal plane of the telescope as seen by an observer looking southwards at it, the cœlostat being to the east of the axis of the telescope.

To sum up, it may be noted that with a cœlostat arranged in this manner, the effect of rotating the secondary mirror about a vertical axis is to cause the image to move in a direction which is inclined, both to the horizontal and to the direction of an east and west line in the image, by an amount which varies during the year. Further, with this instrument the magnitude of the angular velocity of the image is always less than twice the velocity of rotation of the secondary mirror, the numerical factor again varying with the Sun's declination.

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RADIAL VELOCITIES AND STAR-STREAMING.

W. M. Smart, M.A., D.Sc.

(Communicated by the Director, Cambridge Observatory)

1. It has not hitherto been found practicable to make use of the radial velocities of the stars in deriving an independent determination of the drift constants on the two-streams theory, but it is, nevertheless, important to ascertain how closely the observed radial velocities are in accordance with this theory. Eddington * has made one such test, the method being to compare the mean radial velocity (freed from solar motion) in a region of the sky in the direction of the vertex of star-streaming with the mean radial velocity (also freed from the solar motion) in a region 90° away from the vertex. The theoretical ratio can also be found in terms of the drift constants, and, with the somewhat meagre amount of observational material available at the time, the accordance was reasonably satisfactory. With the great increase in radial velocity measures it has seemed worth while to undertake a comprehensive investigation to include radial velocities in all parts of the sky. The radial velocities were taken from Schlesinger's *Catalogue of Bright Stars*; as the limiting visual magnitude in this catalogue is $6^{m}\cdot5$, most of the stars with radial velocity determinations belong also to Boss's

* *Stellar Movements*, p. 144, 1914.

P.G.C., the proper motions of which were used by Eddington * in analysing the systematic motions of the stars. Accordingly, in this paper I have used Eddington's values of the drift constants. Stars omitted from consideration are (*a*) stars of spectral type other than B to M, (*b*) stars belonging to open clusters and (*c*) stars in Oort's category † of "high velocity stars." I am indebted to Mr. H. E. Green for assistance in preparing the material for the present investigation.

2. The method alluded to in the previous section is rather laborious in its general application, as the line-of-sight component of the solar motion has to be found for each star—or at least for the centre of each region considered. The method of this paper is based on a formula which I recently gave,‡ and it involves the *observed* radial velocities directly.

Consider a drift of stars in which the individual random radial velocities, R, for a small region of the sky are distributed according to the law $G(R)dR$. Let $\bar{R}$ denote the mean peculiar radial velocity and $\bar{R}_0'$ the mean *observed* radial velocity, both taken without regard to sign. It is shown (*loc. cit.*) that, if V is the component, in the line of sight, of the drift motion with respect to the Sun,

$$\bar{R}_0' = \bar{R} + \frac{1}{Q}\int_0^V (V - R)G(R)dR,$$

where

$$Q = \int_0^\infty G(R)dR.$$

Taking, as usual, $G(R)$ to be a Maxwellian function so that

$$G(R) = e^{-h^2R^2},$$

we find that

$$Q = \frac{\sqrt{\pi}}{2h} = \frac{1}{2h^2\bar{R}},$$

since $\bar{R} = \dfrac{1}{h\sqrt{\pi}}$, and

$$\bar{R}_0' = \bar{R}\left\{1 + 2h^2\int_0^V (V - R)e^{-h^2R^2}dR\right\},$$

which is easily seen to reduce to

$$\bar{R}_0' = \bar{R}\left\{2hV\int_0^{hV} e^{-x^2}dx + e^{-h^2V^2}\right\}.$$

If the drift velocity is U and λ is the angular distance of the region from the drift apex, we have $V = U\cos\lambda$.

We write

$$\tau = hV = hU\cos\lambda,$$

$$\Theta(\tau) = \frac{2}{\sqrt{\pi}}\int_0^\tau e^{-x^2}dx.$$

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* *M.N.*, **71**, 4, 1910. † *Groningen Publications*, **40**, 30, 1926.
‡ *M.N.*, **95**, 122, 1934 (formula 35).

Then

$$\bar{R}_0' = \bar{R}F(\tau),$$

where

$$F(\tau) = \sqrt{\pi}\,\tau\,\Theta(\tau) + e^{-\tau^2}.$$

$F(\tau)$ is an even function of τ, and its values can be readily computed with the help of the table of $\Theta(\tau)$ given in Brunt's *Combination of Observations*, Appendix II, 1923. The values of $F(\tau)$ for a sufficient range in τ are shown in Table I.

TABLE I

Values of the Function $F(\tau)$

τ	$F(\tau)$	τ	$F(\tau)$	τ	$F(\tau)$	τ	$F(\tau)$
0·00	1·000	0·50	1·240	1·00	1·861	1·50	2·673
0·05	1·003	0·55	1·288	1·05	1·936	1·55	2·759
0·10	1·010	0·60	1·340	1·10	2·014	1·60	2·845
0·15	1·022	0·65	1·395	1·15	2·093	1·65	2·932
0·20	1·040	0·70	1·453	1·20	2·173	1·70	3·019
0·25	1·062	0·75	1·515	1·25	2·254	1·75	3·106
0·30	1·089	0·80	1·579	1·30	2·336	1·80	3·194
0·35	1·120	0·85	1·646	1·35	2·419	1·85	3·282
0·40	1·156	0·90	1·716	1·40	2·503	1·90	3·370
0·45	1·196	0·95	1·787	1·45	2·588	1·95	3·458
0·50	1·240	1·00	1·861	1·50	2·673	2·00	3·546

Let now hU_1 and hU_2 be the velocities of drift I and drift II, λ_1 and λ_2 the angular distances of the region considered from the drift apices, and α and $(1-\alpha)$ the proportions of stars in the two drifts. Denoting by $\bar{R}_0$ the mean observed radial velocity, *taken regardless of sign*, of all the stars in the region, we have

$$\bar{R}_0 = \bar{R}\{\alpha F(\tau_1) + (1-\alpha)F(\tau_2)\}, \qquad (1)$$

where

$$\tau_1 = hU_1 \cos\lambda_1 \quad \text{and} \quad \tau_2 = hU_2 \cos\lambda_2. \qquad (2)$$

We adopt Eddington's results, namely,

$$hU_1 = 1{\cdot}516 \quad \text{and} \quad hU_2 = 0{\cdot}855.$$

The values of λ_1 and λ_2 are also computed for Eddington's co-ordinates of the apices.

The radial velocities of the preceding formulæ are all expressed in terms of the theoretical unit $1/h$. In terms of this unit, Eddington's value for the solar motion is 0·908, and identifying this with the usual value of 19·5 km./sec., we easily find that

$$\bar{R} \equiv \frac{1}{h\sqrt{\pi}} = 12{\cdot}13 \text{ km./sec.}$$

The analysis of the proper motions of the Boss stars gave $\alpha = 0{\cdot}6$ approximately; for fainter stars it is usually found that there is almost an equal division between the two drifts so that $\alpha = 0{\cdot}5$. With either of these values of α, the right-hand side of (1) can now be computed, in km./sec., for each region of the sky, and the result compared with the corresponding observed

value of $\bar{R}_0$. On the two-streams theory these results should be accordant for each region. When the calculations were made this expectation was realised very closely in the case of $\alpha = 0{\cdot}5$. I then decided to regard (1) as an observational equation with two unknowns, $\bar{R}$ and α, to be found by a least squares solution, the values of $\bar{R}_0$ being given by the observations.

Write

$$\bar{R} = x, \qquad \alpha = 0{\cdot}5 + \beta, \qquad \beta\bar{R} = y,$$

and put

$$\tfrac{1}{2}F(\tau_1) + \tfrac{1}{2}F(\tau_2) = a, \qquad F(\tau_1) - F(\tau_2) = b.$$

Then (1) becomes

$$ax + by = \bar{R}_0, \tag{3}$$

in which a, b and $\bar{R}_0$ are known for each region.

TABLE II

Region	Gal. Long.	Gal. Lat.	$\Sigma \mid R_0 \mid$ according to Spectral Type					
			B	A	F	G	K	M
H	112½°	0°	718	444	529	319	765	104
			(64)	(38)	(31)	(17)	(51)	(6)
I	157½	0	874	654	453	244	1230	288
			(51)	(35)	(21)	(13)	(51)	(9)
J	202½	0	1436	779	887	384	1425	221
			(71)	(40)	(38)	(26)	(65)	(9)
K	247½	0	1751	623	454	278	1088	167
			(95)	(39)	(30)	(19)	(71)	(10)
E	120	+20	295	373	347	411	1313	245
			(20)	(37)	(21)	(24)	(64)	(12)
F	180	+20	370	523	574	655	1447	191
			(19)	(35)	(28)	(32)	(64)	(10)
G	240	+20	395	428	489	284	1100	152
			(30)	(36)	(31)	(21)	(54)	(7)
L	120	−20	693	477	341	378	1265	154
			(70)	(43)	(24)	(23)	(63)	(8)
M	180	−20	1867	1070	458	507	1140	240
			(77)	(57)	(21)	(18)	(52)	(9)
N	240	−20	782	745	660	367	1593	280
			(37)	(45)	(38)	(24)	(72)	(12)
B	120	+43	112	364	549	528	1395	173
			(10)	(43)	(33)	(33)	(70)	(12)
C	180	+43	214	599	389	693	1434	312
			(17)	(44)	(27)	(31)	(71)	(17)
D	240	+43	153	272	288	451	872	250
			(18)	(27)	(26)	(24)	(59)	(21)
P	120	−43	256	338	536	637	1296	421
			(20)	(35)	(39)	(33)	(67)	(15)
Q	180	−43	327	642	325	598	1439	383
			(20)	(42)	(17)	(25)	(63)	(16)
R	240	−43	139	497	453	512	1140	338
			(11)	(44)	(32)	(24)	(50)	(15)
A	180	+69	181	953	889	629	2154	508
			(15)	(102)	(64)	(46)	(129)	(34)

3. The sky has been divided up into the 36 areas, according to galactic latitude, used in R. D. H. Jones's discussion * of the Boss proper motions. The details of the areas will be found in the paper referred to. A region is formed by taking together two antipodal areas. In Table II are the regions (with the exception of *S*) according to Jones's designation ; in the second and third columns are the galactic co-ordinates of the centre of one of each pair of constituent areas, and in the remaining columns are the observed values of $\Sigma \mid R_0 \mid$, in km./sec., with the number of stars in parentheses, arranged according to spectral type. Eddington's values of the drift apices in galactic co-ordinates are (189°, −15°) for drift I and (299½°, −28°) for drift II ; the values of the angular distances, λ_1 and λ_2, of the centres of each region from the drift apices are then calculated in the usual way. Table III contains the values of λ_1, λ_2 ; τ_1, τ_2 ; and $F(\tau_1)$, $F(\tau_2)$—the latter obtained from Table I.

Table III

Region	λ_1	λ_2	τ_1	τ_2	$F(\tau_1)$	$F(\tau_2)$
	°	°				
H	77·0	151·2	0·341	−0·750	1·115	1·515
I	34·5	134·1	1·250	−0·600	2·254	1·340
J	20·1	96·2	1·425	−0·090	2·545	1·010
K	59·7	57·1	0·765	+0·466	1·534	1·211
E	76·3	171·4	0·358	−0·846	1·126	1·640
F	36·0	124·7	1·226	−0·486	2·215	1·228
G	61·1	74·9	0·731	+0·221	1·490	1·045
L	65·5	131·9	0·628	−0·571	1·371	1·310
M	9·5	104·4	1·494	−0·213	2·663	1·046
N	48·7	54·5	1·000	+0·496	1·861	1·237
B	85·6	165·0	0·119	−0·826	1·015	1·613
C	58·5	129·7	0·791	−0·546	1·567	1·284
D	74·5	89·6	0·405	+0·020	1·160	1·001
P	64·5	109·0	0·652	−0·278	1·397	1·077
Q	28·9	89·9	1·326	0·000	2·378	1·000
R	51·6	49·6	0·941	+0·555	1·773	1·293
A	84·2	126·5	0·155	−0·508	1·024	1·248

4. Two solutions of the equations of type (3) have been made ; in the first, denoted by (i), all the data of Table II concerning 3679 stars has been utilised, while in the second solution, denoted by (ii), concerning 3034 stars the B-type stars have been omitted from the statistics. The details for the solution of (3) are given in Table IV. It is to be remembered that $\bar{R}_0$ in the second column of Table IV is the mean radial velocity, regardless of sign, of the stars of spectral types B to M, whereas in the third column it is the mean for types A to M. In proper motion investigations of the bright stars it has generally been found that B-type stars show the streaming phenomenon to a very limited extent ; this may be partly due to the fact that, owing to their great distances, these stars have, in the main, small proper motions, and the effect of accidental errors in the latter will consequently be a tendency to conceal the separation into two drifts. It was hoped that the solutions (i) and (ii) would throw some light on this point.

* "Star-streaming in Galactic Zones," *M.N.*, **91**, 564, 1931.

TABLE IV

Region	$\bar{R}_0$ (km./sec.)		No. of Stars		a	b	Residuals (km./sec.)	
	(i)	(ii)	(i)	(ii)			(i)	(ii)
H	13·9	15·1	207	143	1·32	−0·40	+1·5	+1·1
I	20·8	22·1	180	129	1·80	+0·91	+0·6	−0·3
J	20·6	20·8	249	178	1·78	1·53	+0·8	+0·7
K	16·5	15·5	264	169	1·37	+0·32	−0·2	+1·2
E	16·8	17·1	178	158	1·38	−0·51	−0·6	−0·2
F	20·0	20·1	188	169	1·72	+0·99	+0·5	−0·7
G	15·9	16·5	179	149	1·27	0·45	−0·8	−1·1
L	14·3	16·2	231	161	1·34	0·06	+1·5	+0·1
M	22·6	21·8	234	157	1·85	1·62	−0·4	+0·6
N	19·4	19·1	228	191	1·55	+0·62	−0·9	−0·3
B	15·5	15·7	201	191	1·31	−0·60	−0·1	+0·4
C	17·6	18·0	207	190	1·42	+0·28	−0·7	−0·7
D	13·1	13·6	175	157	1·08	0·16	−0·4	−0·4
P	16·7	17·1	209	189	1·24	0·32	−2·0	−2·0
Q	20·3	20·8	183	163	1·69	1·38	−0·1	−0·4
R	17·5	17·9	176	165	1·53	+0·48	+0·7	+0·7
A	13·6	13·7	390	375	1·14	−0·22	−0·2	+0·2

Except for region A, Table IV shows that the numbers of stars are fairly constant from one region to another, and, in consequence, I have given each region unit weight in forming the normal equations from (3). The latter are as follows :—

	(i)	(ii)
$37·06x + 12·88y = 440·6$;		449·0
$12·88x + 10·63y = 154·6$;		155·1

The solutions are :

$x =$	11·81 ;	12·17
$y = +$	0·234 ;	− 0·155

Hence we obtain $\bar{R}$ (in km./sec.) and a, with their probable errors:

	(i)	(ii)
$\bar{R} =$	**11·81** ± 0·135 ;	**12·17** ± 0·124
$a =$	**0·520** ± 0·022 ;	**0·487** ± 0·020

The residuals for each region are shown in the last two columns of Table IV.

The results show two noteworthy features: first, the value of $\bar{R}$ in solution (ii) is almost exactly that deduced in section 2 from Eddington's value of the solar velocity expressed in terms of the theoretical unit $1/h$, and the value of the solar velocity, in km./sec., found in the ordinary way from radial velocity measures ; and second, the stars are divided practically equally between the two streams.

The second solution, which is based on considerably fewer stars than the first solution, leads to rather smaller probable errors, and it would seem—on the present evidence—that the B-type stars do tend, to a small extent at least, to conceal the chief features of streaming.

As regards the principal object of this investigation, the residuals in the last two columns of Table IV show unmistakably that the radial velocities over the whole sky are in excellent accordance with the two-streams theory.

5. I have also made solutions for the regions taken according to galactic latitude. As before, (i) denotes that all spectral types from B to M are included, and (ii) that the B stars are omitted. The results are in Table V, the probable errors being given in parentheses.

TABLE V

Solutions according to Galactic Latitude

Gal. Lat.	No. of Stars		$\bar{R}$ (km./sec.)		a	
	(i)	(ii)	(i)	(ii)	(i)	(ii)
°						
0	900	619	11·16 (± 0·26)	11·62 (± 0·24)	0·567 (± 0·040)	0·538 (± 0·035)
20	1238	985	11·83 (± 0·25)	12·34 (± 0·17)	0·531 (± 0·037)	0·454 (± 0·024)
43	1151	1055	12·13 (± 0·29)	12·40 (± 0·26)	0·503 (± 0·050)	0·509 (± 0·040)

Both solutions show a progression of $\bar{R}$ with galactic latitude. There appear to be two alternative explanations. (*a*) Since $\bar{R} = 1/h\sqrt{\pi}$, in theoretical units, a progression of $\bar{R}$ (in km./sec.) with galactic latitude corresponds to a progression of $1/h$ when expressed in km./sec. If the solar speed, V (in km./sec.) is invariable for all galactic latitudes, the theoretical value hV (which can be written $V \div 1/h$ in terms of the unit $1/h$) consequently decreases with galactic latitude. In this case each drift of stars must be supposed to be composed of strata parallel to the galactic equator, with greater freedom of movement as we get further from this plane. (*b*) If the stars at all galactic latitudes form two kinematically isotropic drifts, the apparent progression in $\bar{R}$ must be counterbalanced by a decrease in $1/h$, when expressed in km./sec., with increasing galactic latitude. Assuming that the solar speed (in km./sec.) is constant for all galactic latitudes, it follows that its value (in terms of $1/h$) increases with galactic latitude.

R. D. H. Jones found some evidence in favour of the first explanation from the analysis * of the Boss proper motions according to galactic latitude; although the solar speed (in terms of $1/h$) was practically the same for latitudes 0° and 20°, there was a sharp drop in its value for latitude 43°. For this reason the second explanation, I think, must be ruled out.

6. *Summary.*

(i) The radial velocities of 3679 stars (arranged in galactic zones) are used to test the two-streams theory.

(ii) The mean radial speed of the stars in a drift is found to be almost exactly that deduced from Eddington's analysis of the Boss proper motions.

(iii) The stars are equally divided between the two drifts.

(iv) A small progression, with galactic latitude, of the mean radial speed of the stars in a drift is found.

Observatory, Cambridge:
1935 *January* 2.

* *M.N.*, **91**, 568, 1931.

ON THE DERIVATION OF THE FREQUENCY FUNCTION OF SPACE VELOCITIES OF THE STARS FROM THE OBSERVED RADIAL VELOCITIES.

V. Ambarzumian.

(Communicated by Sir Arthur Eddington)

One of the most important problems of stellar statistics is the derivation of the frequency function of the space velocities of stars of various spectral types and of different absolute magnitudes in our neighbourhood. The direct solution of this problem requires the knowledge of the space velocities of a great number of stars. The derivation of the space velocity of a given star is possible only in the case when three different quantities are measured : the radial velocity, the proper motion and the parallax. These quantities are measurable with different relative degrees of accuracy and are exposed to systematic errors of quite different kinds. For some important groups of stars (for example B-type stars) we have very few reliable individual parallaxes. In general, the number of reliable parallaxes is generally small, and comparatively few stars with known radial velocities have known parallaxes.

Therefore several writers have made the attempts to obtain some knowledge about the distribution law of space velocities from the radial velocities alone. However, in every case some more or less arbitrary form of this law was assumed, and the problem was restricted to the finding of numerical values of some constant parameters entering in this form of distribution law. In the majority of cases these constants are the elements of the velocity ellipsoids.

Owing to the relative uniformity of the catalogues of the radial velocities, the results of the statistical investigations based on them are almost free from the influence of systematic errors.

It seems desirable, then, to try to solve the problem of derivation of the frequency function of space velocities from the distribution of radial velocities without making any hypothesis about the form of this function.

So far as it is known to the writer, this problem not only remains yet unsolved, but is not even discussed in any detail. The purpose of the present paper is to derive the general formula which enables us to compute the frequency function of space velocities from the distribution of radial velocities.

It will be shown that the frequency function of the space velocities is the solution of an integral equation. In this equation the observed frequency function of the radial velocities for the different parts of the sky enters as the known function. We give below the derivation of the equation and its solution.

The Fundamental Assumption.—We shall assume that the different

elementary volumes of space in our neighbourhood have practically identical frequency functions of the space velocities. In actual cases, when relatively rare types of stars (for example Cepheids) are considered, it is necessary to consider also the distant stars, since the number of stars of such types in our neighbourhood is very small. In such cases some corrections for the difference between the frequency functions in various parts of the galaxy are required. The actual process of introducing these corrections is beyond the scope of the present paper. We suppose that the radial velocities of a sufficiently large number of near stars situated in different parts of the sky are given, and our aim is to derive the frequency function of the space velocities from these radial velocities.

We shall consider first the two-dimensional problem. It is of special interest, since some types of stars are strongly concentrated to the galactic plane and the z-components of their velocities are small.

The Two-dimensional Problem.—If the stars are distributed over a plane and we are situated in the same plane, we may measure for each star the radial velocity V and its apparent position or azimuth, reckoned from some fixed direction. In the case of the stars of high galactic concentration the rôle of such azimuth is played by the galactic longitude. Let $f(V, \alpha)dVd\alpha$ be the number of the observed stars with azimuths between α and $\alpha+d\alpha$ and with radial velocities between V and $V+dV$. The function $f(V, \alpha)$ is to be obtained from the lists of the radial velocity stars. If, further, $n(\alpha)d\alpha$ is the total number of the observed stars in the directions between α and $\alpha+d\alpha$, we have

$$n(\alpha)=\int_{-\infty}^{\infty} f(V, \alpha)dV.$$

Let $\phi(\xi, \eta)$ be the unknown frequency function of true velocities. According to the definition, $\phi(\xi, \eta)d\xi d\eta$ is the relative number of stars for which the velocity components fall within the limits ξ and $\xi+d\xi$, η and $\eta+d\eta$. We have

$$\int_{-\infty}^{\infty} d\eta \int_{-\infty}^{\infty} \phi(\xi, \eta)d\xi = 1.$$

Among the observed stars with the azimuths confined within the interval $(\alpha, \alpha+d\alpha)$ we have $n(\alpha)d\alpha\phi(\xi, \eta)d\xi d\eta$ stars, which have the velocities confined within the element $d\xi d\eta$ of the "velocity plane" $\xi\eta$.

We may direct the ξ-axis to the azimuth $\alpha=0$. Then it is clear that all stars observed in the azimuth α for which the velocities lie within the strip S of $\xi\eta$ plane (see fig.) have the radial velocities lying between V and $V+dV$. Therefore from $n(\alpha)d\alpha$ stars within the interval $(\alpha, \alpha+d\alpha)$,

$$n(\alpha)d\alpha \int_{(S)} \phi(\xi, \eta)d\xi d\eta$$

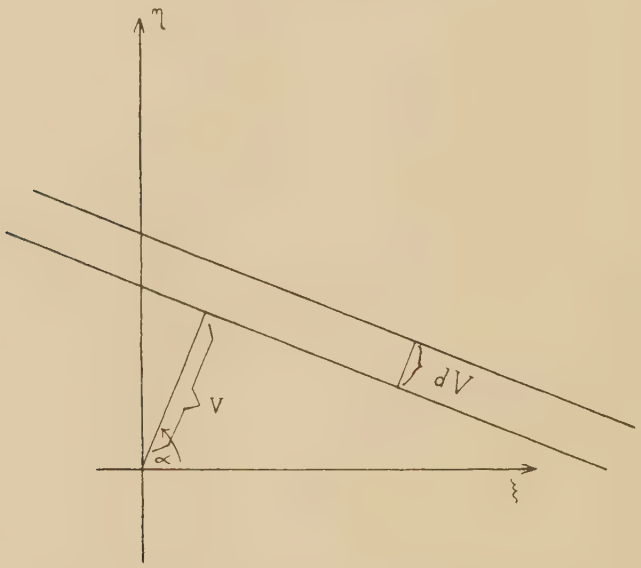
stars will have the radial velocities confined between V and $V+dV$. The integration is carried over the strip (S), perpendicular to the direction α and with the width dV.

On the other hand, we have denoted the number of such stars as

$$f(V, \alpha)dVd\alpha.$$

Therefore we have the equation

$$f(V, \alpha)dV = n(\alpha)\int_{(S)} \phi(\xi, \eta)d\xi d\eta. \tag{1}$$



Let us introduce under the sign of the integral instead of ξ and η the new co-ordinates :

$$\xi' = \xi \cos \alpha + \eta \sin \alpha,$$
$$\eta' = -\xi \sin \alpha + \eta \cos \alpha.$$

It is clear that ξ' within the strip (S) varies between V and $V+dV$, and η' varies between $-\infty$ and $+\infty$. Therefore

$$\int_{(S)} \phi(\xi, \eta)d\xi d\eta = \int_{V}^{V+dV} d\xi' \int_{-\infty}^{\infty} \phi(\xi' \cos \alpha - \eta' \sin \alpha, \xi' \sin \alpha + \eta' \cos \alpha)d\eta'.$$

If we divide the equation (1) by $n(\alpha)$ and set

$$F(V, \alpha) = \frac{f(V, \alpha)}{n(\alpha)} = \frac{f(V, \alpha)}{\int f(V, \alpha)dV},$$

we obtain the equation

$$F(V, \alpha) = \int_{-\infty}^{\infty} \phi(V \cos \alpha - \eta' \sin \alpha, V \sin \alpha + \eta' \cos \alpha)d\eta'. \tag{2}$$

The left-hand side of this equation may be obtained from the counts of stars in the radial-velocity lists.

Returning to the old co-ordinates ξ and η, we may write the equation (2) in the form

$$F(V, \alpha) = \int_{(L)} \phi(\xi, \eta) ds, \tag{3}$$

where the integration is carried over the straight line (L) defined by mean of equation

$$\xi \cos \alpha + \eta \sin \alpha = V, \tag{4}$$

and ds is the element of this line.

The equation (3) expresses the following problem :—

The value of the integral (3) *for every straight line of the* $\xi\eta$ *plane is given as the function of the parameters* V *and* α, *defining the straight line. The integrand* $\phi(\xi, \eta)$ *is to be found.*

The solution of this problem is comparatively simple. Let us introduce in both parts of (2) instead of V the expression

$$V = x \cos \alpha + y \sin \alpha + W, \tag{5}$$

where x, y and W are some arbitrary parameters. Then we can rewrite (2) in the form

$$F(x \cos \alpha + y \sin \alpha + W, \alpha) = \int_{-\infty}^{\infty} \phi(x \cos^2 \alpha + y \sin \alpha \cos \alpha + W \cos \alpha - \eta' \sin \alpha, \\ x \cos \alpha \sin \alpha + y^2 \sin^2 \alpha + W \sin \alpha + \eta' \cos \alpha) d\eta'.$$

If we introduce the new variable of integration

$$\eta' = U - x \sin \alpha + y \cos \alpha, \tag{6}$$

our equation takes the simple form

$$F(x \cos \alpha + y \sin \alpha + W, \alpha) = \int_{-\infty}^{\infty} \phi(x + W \cos \alpha - U \sin \alpha, \\ y + W \sin \alpha + U \cos \alpha) dU. \tag{7}$$

Multiplying both parts of this equation by $d\alpha$, integrating between 0 and 2π and changing the order of the integration on the right-hand side, we find

$$\int_0^{2\pi} F(x \cos \alpha + y \sin \alpha + W, \alpha) = \int_{-\infty}^{\infty} dU \int_0^{2\pi} \phi(x + W \cos \alpha - U \sin \alpha, \\ y + W \sin \alpha + U \cos \alpha) d\alpha. \tag{8}$$

Now it is easy to see that the integral

$$\Phi = \int_0^{2\pi} \phi(x + W \cos \alpha - U \sin \alpha, y + W \sin \alpha + U \cos \alpha) d\alpha, \tag{9}$$

depends only on x, y and $\sqrt{W^2 + U^2}$. In fact, if we introduce in (9)

$$\left.\begin{aligned} W &= G \cos \beta \\ U &= G \sin \beta \end{aligned}\right\}, \tag{10}$$

we obtain

$$\Phi = \int_0^{2\pi} \phi[x + G \cos (\alpha + \beta), y + G \sin (\alpha + \beta)] d\alpha,$$

and it is obvious that this integral depends only on x, y and $G=\sqrt{W^2+U^2}$ and is independent of $\beta \equiv \operatorname{arctg}\frac{U}{W}$.

Therefore we may write simply :

$$\Phi(x, y, G)=\int_0^{2\pi}\phi(x+G\cos\alpha, y+G\sin\alpha)d\alpha \tag{11}$$

and

$$\int_0^{2\pi}F(x\cos\alpha+y\sin\alpha+W, \alpha)d\alpha=\int_{-\infty}^{\infty}\Phi(x, y, G)dU. \tag{12}$$

However,

$$dU=\frac{GdG}{\sqrt{G^2-W^2}},$$

and we may rewrite (12) in the form

$$\int_0^{2\pi}F(x\cos\alpha+y\sin\alpha+W, \alpha)d\alpha=2\int_W^{\infty}\Phi(x, y, G)\frac{GdG}{\sqrt{G^2-W^2}}. \tag{13}$$

This equation is an integral equation of Abel's type for the function $\Phi(x, y, G)$, and its solution is given by

$$\Phi(x, y, G)=-\frac{1}{\pi}\frac{1}{G}\frac{d}{dG}\int_G^{\infty}\frac{WdW}{\sqrt{W^2-G^2}}\int_0^{2\pi}F(x\cos\alpha+y\sin\alpha+W, \alpha)d\alpha. \tag{14}$$

We have, according to (11),

$$\Phi(x, y, 0)=2\pi\phi(x, y), \tag{15}$$

and we may rewrite (14) in the form

$$\phi(x, y)=-\frac{1}{2\pi^2}\lim_{G\to 0}\frac{1}{G}\frac{d}{dG}\int_G^{\infty}\frac{WdW}{\sqrt{W^2-G^2}}\bar{F}(x, y, W), \tag{16}$$

where the function

$$\bar{F}(x, y, W)=\int_0^{2\pi}F(x\cos\alpha+y\sin\alpha+W, \alpha)d\alpha \tag{17}$$

may be obtained from the observations. After some algebra we can reduce (16) to the form

$$\phi(x, y)=-\frac{1}{2\pi^2}\int_0^{\infty}\frac{1}{W}\frac{d\bar{F}(x, y, W)}{dW}dW. \tag{18}$$

This equation gives us the solution of our problem. The numerical calculation of $\bar{F}(x, y, W)$, when $F(V, \alpha)$ is given, may be carried out without difficulty.

We have actually applied our formulæ to the radial velocities of B-type stars observed in the galactic belt $\xi \mid b \mid < 20°$, and the results of this application are in satisfactory agreement with the velocity distribution derived from the direct counts of the known space velocities.

The details of this application will be given elsewhere.

The Three-dimensional Problem.—In the case of the three-dimensional problem we may derive from the catalogues the number of stars observed within the given solid angle $d\omega$ in the given direction, having the radial velocity confined within the limits V and $V+dV$. Let us denote this number by $f(V, l, b)d\omega$, where l and b are galactic longitude and latitude. If, further, $n(l, b)d\omega$ is the total number of observed stars in the same solid angle, we shall have

$$n(l, b) = \int f(V, l, b)dV. \tag{19}$$

As in the case of the two-dimensional problem we have the following relation between the frequency function of the space velocities $\phi(\xi, \eta, \zeta)$ and the observed function $f(V, l, b)$:—

$$f(V, l, b)dV = n(l, b)\iiint_{(\Omega)} \phi(\xi, \eta, \zeta)d\xi d\eta d\zeta, \tag{20}$$

where the integration is extended over the volume (Ω) of the $\xi\eta\zeta$ space confined between two parallel planes, which are perpendicular to the direction (l, b) and have the distances V and $V+dV$ from the origin.

Dividing (20) by $n(l, b)$, we can bring this equation after some transformations to the form

$$F(V, l, b) = \iint_{(\Sigma)} \phi(\xi, \eta, \zeta)d\sigma, \tag{21}$$

where

$$F(V, l, b) = \frac{f(V, l, b)}{n(l, b)}, \tag{22}$$

and the integration is extended on the plane (Σ), perpendicular to the direction (l, b) and distant V from the origin.

The equation of this plane is :

$$\xi \cos l \cos b + \eta \sin l \cos b + \zeta \sin b = V. \tag{Σ}$$

If we introduce the polar co-ordinates ρ and θ in the plane (Σ) with the origin at the point,

$$\xi = V \cos l \cos b\,; \qquad \eta = V \sin l \cos b\,; \qquad \zeta = V \sin b,$$

we shall have for the points of this plane

$$\begin{aligned} \xi &= V \cos l \cos b + \rho(\alpha_1 \cos\theta + \beta_1 \sin\theta), \\ \eta &= V \sin l \cos b + \rho(\alpha_2 \cos\theta + \beta_2 \sin\theta), \\ \zeta &= V \sin b + \rho(\alpha_3 \cos\theta + \beta_3 \sin\theta), \end{aligned}$$

where the coefficients α_1, α_2, α_3 and β_1, β_2, β_3 satisfy to the conditions

$$\begin{gathered} \alpha_1 \cos l \cos b + \alpha_2 \sin l \cos b + \alpha_3 \sin b = 0, \\ \beta_1 \cos l \cos b + \beta_2 \sin l \cos b + \beta_3 \sin b = 0. \\ \alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3 = 0. \\ \alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1\,; \qquad \beta_1^2 + \beta_2^2 + \beta_3^2 = 1. \end{gathered}$$

Now we can rewrite the equation (21) in these co-ordinates:

$$F(V, l, b) = \int_0^\infty \rho d\rho \int_0^{2\pi} \phi(V \cos l \cos b + \rho(\alpha_1 \cos\theta + \beta_1 \sin\theta),$$
$$V \sin l \cos b + \rho(\alpha_2 \cos\theta + \beta_2 \sin\theta),$$
$$V \sin b + \rho(\alpha_3 \cos\theta + \beta_3 \sin\theta))d\theta.$$

Integrating over all directions and changing the order of the integration on the right-hand side, we obtain

$$\int F(V, l, b)d\omega = \int \Phi \rho d\rho, \tag{23}$$

where

$$\Phi = \int d\omega \int_0^{2\pi} \phi(V \cos l \cos b + \rho(\alpha_1 \cos\theta + \beta_1 \sin\theta),$$
$$V \sin l \cos b + \rho(\alpha_2 \cos\theta + \beta_2 \sin\theta),$$
$$V \sin b + \rho(\alpha_3 \cos\theta + \beta_3 \sin\theta))d\theta.$$

If we introduce the new parameters

$$V = G\gamma_1; \qquad \rho \cos\theta = G\gamma_2; \qquad \rho \sin\theta = G\gamma_3; \qquad \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1,$$

the integral Φ takes the form

$$\Phi = \int d\theta \int \phi(G(\gamma_1 \cos l \cos b + \gamma_2\alpha_1 + \gamma_3\beta_1),$$
$$G(\gamma_1 \sin l \cos b + \gamma_2\alpha_2 + \gamma_3\beta_2),$$
$$G(\gamma_1 \sin b + \gamma_2\alpha_3 + \gamma_3\beta_3))d\omega,$$

and it may be shown that it depends only on G. We may write

$$\Phi(G) = 2\pi \int \phi(G \cos l \cos b, G \sin l \cos b, G \sin b)d\omega; \quad \Phi(0) = 8\pi^2\phi \mid 0, 0, 0) \tag{24}$$

We have now

$$G^2 = V^2 + \rho^2; \qquad \rho d\rho = GdG.$$

Therefore

$$\int F(V, l, b)d\omega = \int_V^\infty \Phi G dG \tag{25}$$

and

$$\Phi = -\frac{1}{V}\frac{d}{dV}\int F(V, l, b)d\omega. \tag{26}$$

Comparing (26) with (27) we find

$$\phi(0, 0, 0) = \frac{1}{8\pi^2}\Phi(0) = -\lim_{V\to 0}\frac{1}{V}\frac{d}{dV}\int F(V, l, b)d\omega.$$

Thus we may find $\phi(0, 0, 0)$. In the same way after some lengthy algebra we obtain

$$\phi(\xi, \eta, \zeta) = -\frac{1}{8\pi^2}\frac{1}{W}\frac{d}{dW}\int F(\xi \cos l \cos b + \eta \sin l \cos b + \zeta \sin b + W, l, b)d\omega.$$

This formula represents the solution of the three-dimensional problem.

Concluding Remarks.—Many details of the above method are not yet discussed. Thus in our method it is supposed that the K-effect is absent, since in the presence of the K-effect the frequency function of the space velocities is different in the various directions. Actually, however, we may determine the K-term from the radial velocities and exclude it.

The discussion of the practical side of the above-given method with the application to some classes of stars will be given elsewhere.

Astronomical Observatory of the University, Leningrad :
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THE PRESSURE IN THE INTERIOR OF A STAR.

E. A. Milne.

1. It has been shown by Eddington * that the minimum value of the central pressure in a star of mass M and radius r_1, provided that the density does not decrease inwards, is

$$\frac{3}{8\pi}\frac{GM^2}{r_1^4},$$

G being the constant of gravitation. It is the object of the present note to prove a somewhat more general form of the theorem by a different method. In Eddington's proof the desired result was established by transferring matter from one shell to another shell, and altering the temperature distribution so that the new configuration is one of equilibrium. Thus the theorem was only shown to hold good for *compressible* matter. Further, this method, the "method of transfer," involves the consideration of *two* equilibrium configurations. It is a general principle that any inequality which can be established for configurations of equilibrium must be an analytic consequence of the equations of equilibrium, and such an analytic proof is here given. It then appears that Eddington's theorem holds good not only when $\rho(r)$, the density at distance r from the centre, is a non-increasing function of r, but also under the less stringent condition that

$$\rho(r) < \bar{\rho}(r),$$

where $\bar{\rho}(r)$ is the mean density of the matter enclosed in the sphere of radius r. Thus outward increase of ρ is permissible provided ρ does not increase beyond the mean density of the matter interior to the point considered. Finally, our proof applies without the restriction to compressible matter—it applies, for example, to matter stratified in incompressible concentric layers of different densities.

2. To illustrate the method of proof, we first establish a theorem of a more elementary character.

* *I.C.S.*, 1926, p. 91.

Theorem I.—In any equilibrium configuration the function

$$P+\frac{G[M(r)]^2}{8\pi r^2}$$

decreases outwards, where P is the total pressure at distance r from the centre, $M(r)$ the mass enclosed.

Proof.—By the condition of hydrostatic equilibrium,

$$\frac{dP}{dr}=-\frac{GM(r)}{r^2}\rho. \tag{1}$$

But

$$dM(r)=4\pi r^2\rho dr. \tag{2}$$

Hence

$$\frac{dP}{dr}=-\frac{GM(r)}{4\pi r^4}\frac{dM(r)}{dr}.$$

Hence

$$\frac{d}{dr}\left[P+\frac{GM^2(r)}{8\pi r^4}\right]=-\frac{GM^2(r)}{2\pi r^5}.$$

The right-hand side is essentially negative. Hence the theorem.

Corollary.—It follows that if P_c denotes the central pressure, P_1 the pressure at the conventionally assigned boundary, where the radius is r_1 and mass enclosed M, then

$$P_c>P+\frac{GM^2(r)}{8\pi r^4}>P_1+\frac{GM^2}{8\pi r_1^4}. \tag{3}$$

The outside members of this inequality provide a lower limit for the central pressure in any configuration of equilibrium, whatever the law of density.

3. We proceed to the main theorem.

Theorem II.—In any equilibrium configuration in which $\bar{\rho}(r)$, the mean density inside r, always exceeds $\rho(r)$, the actual density at r, the function

$$P+\frac{3}{8\pi}\frac{G[M(r)]^2}{r^4}$$

decreases outwards.

Proof.—Consider the expression

$$P+\alpha\frac{GM^2(r)}{8\pi r^4},$$

where α is an arbitrary number.

We have

$$\frac{d}{dr}\left[P+\alpha\frac{GM^2(r)}{8\pi r^4}\right]=\frac{dP}{dr}+\alpha\frac{GM(r)}{4\pi r^4}\frac{dM(r)}{dr}-\alpha\frac{GM^2(r)}{2\pi r^5}.$$

Using (1) and (2) this gives

$$\frac{d}{dr}\left[P+\alpha\frac{GM^2(r)}{8\pi r^4}\right]=\frac{GM(r)}{r^2}\left[(\alpha-1)\rho-\frac{2\alpha}{3}\cdot\frac{M(r)}{(4\pi/3)r^3}\right]=\frac{GM(r)}{r^2}\left[(\alpha-1)\rho-\frac{2\alpha}{3}\bar{\rho}(r)\right].$$

This is accordingly negative, or at most zero, provided that

$$\frac{\rho(r)}{\bar{\rho}(r)} \leqslant \frac{2\alpha}{3(\alpha-1)}.$$

We are given that

$$\rho(r) \leqslant \bar{\rho}(r).$$

Hence the condition is satisfied if we take $\alpha = 3$, and this, moreover, gives the best possible result. Hence the theorem. (Since P is known to decrease outwards, the best theorem is obtained by taking the largest possible value for α.)

Corollary.—As before, we have under the conditions of Theorem II,

$$P_c \geqslant P + \frac{3}{8\pi}\frac{GM^2(r)}{r^4} \geqslant P_1 + \frac{3}{8\pi}\frac{GM^2}{r_1^4}. \tag{4}$$

The outside members of this inequality contain Eddington's theorem under more general conditions. It is known, as Eddington pointed out, that

$$P_1 + \frac{3}{8\pi}\frac{GM^2}{r_1^4}$$

is the actual central pressure in a *homogeneous* configuration exposed to boundary pressure P_1. In this case $\rho(r) = \bar{\rho}(r)$.

4. *Applications.*—These inequalities may now be applied to the integral theorem given by the author in a previous paper.* The result is to obtain improved inequalities for the gravitational potential energy and mean value of gravity in any star. Retaining a non-zero surface-pressure, the theorem in question states that

$$\int_0^{r_1}\frac{GM(r)dM(r)}{r^\nu} = 4\pi(4-\nu)\int_0^{r_1}(P-P_1)r^{3-\nu}dr, \tag{5}$$

where ν is any number less than 4. This is an immediate consequence of (1) and (2). Denote the integral on the left-hand side of (5) by I_ν.

5. Under the conditions of Theorem I we have, on applying (3) to the right-hand side of (5),

$$4\pi(4-\nu)\int_0^{r_1}\left[P_c - \frac{GM^2(r)}{8\pi r^4} - P_1\right]r^{3-\nu}dr > I_\nu > 4\pi(4-\nu)\int_0^{r_1}\left[\frac{GM^2}{8\pi r_1^4} - \frac{GM^2(r)}{8\pi r^4}\right]r^{3-\nu}dr$$

or

$$4\pi(P_c - P_1)r_1^{4-\nu} > I_\nu + \frac{4-\nu}{2}\int_0^{r_1}\frac{GM^2(r)}{r^{1+\nu}}dr > \frac{GM^2}{2r_1^\nu}. \tag{6}$$

But

$$\int_0^{r_1}\frac{GM^2(r)}{r^{1+\nu}}dr = \left[-\frac{GM^2(r)}{\nu r^\nu}\right]_0^{r_1} + \frac{2}{\nu}\int_0^{r_1}\frac{GM(r)dM(r)}{r^\nu} = -\frac{GM^2}{\nu r_1^\nu} + \frac{2}{\nu}I_\nu. \tag{7}$$

Inserting (7) in (6) we find after rearrangement

$$\frac{\nu}{\nu+2}4\pi(P_c - P_1)r_1^{4-\nu} + \frac{GM^2}{(\nu+2)r_1^\nu} > I_\nu > \frac{GM^2}{2r_1^\nu}. \tag{8}$$

* "Integral Theorems on the Equilibrium of a Star," *M.N.*, 89, 739, 1929.

The (negative) gravitational energy Ω is given by taking $\nu = 1$,

$$\Omega = I_1, \tag{9}$$

and the mean value of the acceleration due to gravity, $\bar{g}$, is given by taking $\nu = 2$,

$$M\bar{g} = I_2. \tag{10}$$

Accordingly by (6) and (8), for $\nu = 1$,

$$4\pi(P_c - P_1)r_1^3 > \Omega + \frac{3}{2}\int_0^{r_1} \frac{GM^2(r)}{r^2}dr > \frac{GM^2}{2r_1} \tag{11}$$

and

$$\frac{4\pi}{3}(P_c - P_1)r_1^3 + \frac{GM^2}{3r_1} > \Omega > \frac{GM^2}{2r_1}. \tag{12}$$

These include the less precise inequality

$$4\pi(P_c - P_1)r_1^3 > \Omega > \frac{GM^2}{2r_1}, \tag{13}$$

which is itself an improvement on the inequality (21*a*), p. 743, of the paper cited.

For $\nu = 2$ we find

$$4\pi(P_c - P_1)r_1^2 > M\bar{g} + \int_0^{r_1} \frac{GM^2(r)}{r^3}dr > \frac{GM^2}{2r_1^2}, \tag{14}$$

and

$$2\pi(P_c - P_1)r_1^2 + \frac{GM^2}{4r_1^2} > M\bar{g} > \frac{GM^2}{2r_1^2}. \tag{15}$$

These include the less precise inequality

$$4\pi(P_c - P_1)r_1^2 > M\bar{g} > \frac{GM^2}{2r_1^2}, \tag{16}$$

which is itself an improvement on the inequality (21*b*), p. 743, of the paper cited. The inequalities of this section hold for any distribution of density.

6. Under the conditions of Theorem II we have, on applying (4) to the right-hand side of (5),

$$4\pi(4-\nu)\int_0^{r_1}\left[P_c - P_1 - \frac{3GM^2(r)}{8\pi r^4}\right]r^{3-\nu}dr \geqslant I_\nu \geqslant 4\pi(4-\nu)\int_0^{r_1}\left[\frac{3GM^2}{8\pi r_1^4} - \frac{3GM^2(r)}{8\pi r^4}\right]r^{3-\nu}dr$$

or

$$4\pi(P_c - P_1)r_1^{4-\nu} \geqslant I_\nu + \frac{3(4-\nu)}{2}\int_0^{r_1}\frac{GM^2(r)}{r^{1+\nu}}dr \geqslant \frac{3}{2}\frac{GM^2}{r_1^\nu}. \tag{6'}$$

Using (7) we find after some re-arrangement,

$$\frac{\nu}{6-\nu}2\pi(P_c - P_1)r_1^{4-\nu} + \frac{3(4-\nu)}{4(6-\nu)}\frac{GM^2}{r_1^\nu} \geqslant I_\nu \geqslant \frac{3}{6-\nu}\frac{GM^2}{r_1^\nu}. \tag{8'}$$

Taking $\nu = 1$ we get from (6′) and (8′)

$$4\pi(P_c - P_1)r_1^3 \geqslant \Omega + \frac{9}{2}\int_0^{r_1}\frac{GM^2(r)}{r^2}dr \geqslant \frac{3}{2}\frac{GM^2}{r_1} \tag{11′}$$

and

$$\frac{2\pi}{5}(P_c - P_1)r_1^3 + \frac{9}{20}\frac{GM^2}{r_1} \geqslant \Omega \geqslant \frac{3}{5}\frac{GM^2}{r_1}. \tag{12′}$$

These include the less precise inequality

$$4\pi(P_c - P_1)r_1^3 \geqslant \Omega \geqslant \frac{3}{5}\frac{GM^2}{r_1}, \tag{13′}$$

which indicates, in comparison with (13), the improvement obtained by making the restrictions of Theorem II. The extreme members of (11′) or (12′) give precisely the same inequality as the extreme members of inequality (4), so that the limits for Ω "correspond" to the limits already obtained for P. The value $3GM^2/5r_1$ for Ω is of course obtained for a homogeneous configuration.

For $\nu = 2$ we get from (6′) and (8′)

$$4\pi(P_c - P_1)r_1^2 \geqslant M\bar{g} + 3\int_0^{r_1}\frac{GM^2(\nu)}{r^3}dr \geqslant \frac{3}{2}\frac{GM^2}{r_1^2} \tag{14′}$$

and

$$\pi(P_c - P_1)r_1^2 + \frac{3}{8}\frac{GM^2}{r_1^2} \geqslant M\bar{g} \geqslant \frac{3}{4}\frac{GM^2}{r_1^2}. \tag{15′}$$

These include the less precise inequality

$$4\pi(P_c - P_1)r_1^2 \geqslant M\bar{g} \geqslant \frac{3}{4}\frac{GM^2}{r_1^2}, \tag{16′}$$

which may be compared with (16). Again we notice that the extreme members of (14′) and (15′) imply the same inequality as the extreme members of (4).

7. The value $\nu = 4$ is excluded from (5). When $\nu = 4$ we get instead, from (1) and (2),

$$P_c - P = \frac{1}{4\pi}\int_0^r \frac{GM(r)dM(r)}{r^4} = \frac{GM^2(r)}{8\pi r^4} + \frac{1}{2\pi}\int_0^r \frac{GM^2(r)}{r^5}dr. \tag{17}$$

This gives the error term in the first of inequalities (3). Relation (17) may be used to furnish an alternative proof of (4) as follows.*

The hypothesis of Theorem II, $\rho(r) < \bar{\rho}(r)$, written in the form

$$M(r) > \tfrac{4}{3}\pi r^3\rho(r),$$

gives

$$M(r)dr > \tfrac{4}{3}\pi r^3\rho(r)dr = \tfrac{1}{3}rdM(r).$$

---

* Some years ago Dr. T. G. Cowling sent me a note containing a device similar to that about to be used.

Hence by (17),

$$P_c - P > \frac{GM^2(r)}{8\pi r^4} + \frac{1}{6\pi}\int_0^r \frac{GM(r)dM(r)}{r^4}$$

or

$$(P_c - P) > \frac{GM^2(r)}{8\pi r^4} + \tfrac{2}{3}(P_c - P).$$

This gives the first of (4). The second of each of (3) and (4) is obtained similarly by integrating from r to r_1 instead of from o to r.

8. I have throughout adopted the unusual course of retaining a non-zero boundary pressure P_1, because I shall show in later papers that important consequences follow by regarding the radius r_1 of the star as that radius to which the material of the star beneath the photosphere is held down by the photospheric pressure P_1. This procedure permits a fairly precise treatment of stellar boundary conditions.

19 *Northmoor Road,*
Oxford:
1935 *December.*

POLYTROPIC EQUILIBRIUM. I. THE RADII OF CONFIGURATIONS UNDER GIVEN EXTERNAL PRESSURE.

E. A. Milne.

1. The purpose of the present paper is to determine the radii of polytropic configurations of given mass M under a given external pressure P_1. Our ultimate object is to apply the results to find the relation between radius and photospheric pressure for actual stars, but in order to separate out the different elements in the problem the present paper is confined to certain properties of polytropes considered in the abstract. The applications to actual stellar conditions will be considered in the two succeeding papers.

2. The problem is to calculate the radius r_1 of a given mass M as a function of P_1, the applied external total pressure, when P, the total pressure at any point, is supposed connected with ρ, the density at the same point, by the polytropic relation

$$P = K\rho^{1+\frac{1}{n}}, \tag{1}$$

K and n being given constants. In actual stars that can be represented by polytropes, K and n may depend on a variety of data, but we begin with the abstract case where K and n are supposed known. The earlier passages of the analysis which follows are well known, but it is necessary to recapitulate them for completeness.

The equation of mechanical equilibrium,

$$\frac{dP}{dr} = -\frac{GM(r)}{r^2}\rho, \tag{2}$$

on using the relation derived from (1),

$$\frac{1}{\rho}\frac{dP}{dr}=(n+1)K\frac{d\rho^{\frac{1}{n}}}{dr},$$

gives

$$\frac{1}{r^2}\frac{d}{dr}\left[r^2\frac{d\rho^{\frac{1}{n}}}{dr}\right]=-\frac{4\pi G}{(n+1)K}\rho. \tag{3}$$

Let ρ_c denote the central density and put

$$\rho=\rho_c\theta^n, \qquad r=\lambda\xi, \tag{4}$$

where

$$\lambda^2=\frac{(n+1)K}{4\pi G\rho_c^{\,1-\frac{1}{n}}}. \tag{5}$$

Then (3) reduces to Emden's equation

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left[\xi^2\frac{d\theta}{d\xi}\right]=-\theta^n. \tag{6}$$

We shall confine attention in this paper, for simplicity, to polytropes in which we can reach the centre * without change of K or n. In that case we require the solution of (6) satisfying Emden's conditions, namely, that at

$$\xi=0, \qquad \theta=1, \qquad \frac{d\theta}{d\xi}=0.$$

Let this solution be determined. Then at any point specified by ξ, where $\theta=\theta(\xi)$, the radius, pressure and mass enclosed are known to be given by

$$r=\left[\frac{(n+1)K}{4\pi G}\right]^{\frac{1}{2}}\rho_c^{\frac{1-n}{2n}}\xi, \tag{7}$$

$$P=K\rho_c^{\frac{1+n}{n}}\theta^{n+1}, \tag{8}$$

$$M(r)=\frac{(n+1)^{\frac{3}{2}}K^{\frac{3}{2}}}{(4\pi)^{\frac{1}{2}}G^{\frac{3}{2}}}\rho_c^{\frac{3-n}{2n}}\left(-\xi^2\frac{d\theta}{d\xi}\right). \tag{9}$$

Let this solution be pursued from the centre outwards until the pressure P decreases to the given value P_1 and the mass enclosed becomes M, the radius r_1. Then

$$r_1=\left[\frac{(n+1)K}{4\pi G}\right]^{\frac{1}{2}}\rho_c^{\frac{1-n}{2n}}\xi_1, \tag{10}$$

$$P_1=K\rho_c^{\frac{1+n}{n}}\theta_1^{n+1}, \tag{11}$$

$$M=\frac{(n+1)^{\frac{3}{2}}K^{\frac{3}{2}}}{(4\pi)^{\frac{1}{2}}G^{\frac{3}{2}}}\rho_c^{\frac{3-n}{2n}}\left(-\xi^2\frac{d\theta}{d\xi}\right)_1, \tag{12}$$

* With exceptions that are obvious all the formulæ remain valid for non-Emden solutions, *i.e.* solutions possessing central singularities. ρ_c has to be replaced by an undetermined multiplier.

where ξ_1, θ_1 are the values of ξ and θ at the boundary, *i.e.* the place where P becomes P_1.

3. Our object is to determine the *observable* properties of the polytrope. The value of K may be supposed determined by measurements of P and ρ at the boundary, n being supposed given *a priori*. But ρ_c is unknown. Eliminating ρ_c by means of (11), we have from (10) and (12)

$$M=\frac{(n+1)^{\frac{3}{2}}K^{\frac{2n}{n+1}}}{(4\pi)^{\frac{1}{2}}G^{\frac{3}{2}}}P_1^{\frac{3-n}{2(1+n)}}\theta_1^{\frac{n-3}{2}}\left(-\xi^2\frac{d\theta}{d\xi}\right)_1, \tag{13}$$

$$r_1=\frac{(n+1)^{\frac{1}{2}}K^{\frac{n}{1+n}}}{(4\pi)^{\frac{1}{2}}G^{\frac{1}{2}}}P_1^{-\frac{n-1}{2(1+n)}}\xi_1\theta_1^{\frac{n-1}{2}}. \tag{14}$$

These contain the solution to the problem: given n, K, P_1 and M, (13) can *in general* be solved to give ξ_1, whence by (14) r_1 is determinate. If we eliminate K by using the relation

$$P_1=K\rho_1^{1+\frac{1}{n}}, \tag{15}$$

we find the relations (to be used later)

$$M=\frac{(n+1)^{\frac{3}{2}}}{(4\pi)^{\frac{1}{2}}G^{\frac{3}{2}}}\frac{P_1^{\frac{3}{2}}}{\rho_1^2}\theta_1^{\frac{n-3}{2}}\left(-\xi^2\frac{d\theta}{d\xi}\right)_1, \tag{16}$$

$$r_1=\frac{(n+1)^{\frac{1}{2}}}{(4\pi)^{\frac{1}{2}}G^{\frac{1}{2}}}\frac{P_1^{\frac{1}{2}}}{\rho_1}\xi_1\theta_1^{\frac{n-1}{2}}, \tag{17}$$

whence the three relations

$$\frac{GM}{r_1}=(n+1)\frac{P_1}{\rho_1}\left(-\frac{\xi}{\theta}\frac{d\theta}{d\xi}\right)_1=(n+1)\frac{P_c}{\rho_c}\left(-\xi\frac{d\theta}{d\xi}\right)_1, \tag{18}$$

$$\frac{M}{4\pi r_1^3}=\rho_1\left[-\frac{1}{\theta^n\xi}\frac{d\theta}{d\xi}\right]_1=\rho_c\left(-\frac{1}{\xi}\frac{d\theta}{d\xi}\right)_1, \tag{19}$$

$$\frac{GM^2}{4\pi r_1^4}=(n+1)P_1\left[\frac{1}{\theta^{n+1}}\left(\frac{d\theta}{d\xi}\right)^2\right]_1=(n+1)P_c\left(\frac{d\theta}{d\xi}\right)_1^2. \tag{20}$$

These relations between the members on the extreme left and right are of course well known, but it is with the first relation in each case that we shall ultimately be concerned.

4. We have now two cases to consider, according as $n=3$ or $n\neq 3$. We shall examine r_1 as a function of P_1 for *fixed* M and K.

Case 1. $n=3$. Then (13) and (14) reduce to

$$M=\frac{4^{\frac{3}{2}}K^{\frac{3}{2}}}{(4\pi)^{\frac{1}{2}}G^{\frac{3}{2}}}\left(-\xi^2\frac{d\theta}{d\xi}\right)_1, \tag{21}$$

$$r_1=\frac{4^{\frac{1}{4}}K^{\frac{3}{4}}}{(4\pi)^{\frac{1}{2}}G^{\frac{1}{4}}}P_1^{-\frac{1}{4}}\xi_1\theta_1. \tag{22}$$

Suppose K is given. Let ξ_0 be the first zero of the Emden function $n=3$. Then since $\xi_1 < \xi_0$, it is known that

$$\left(-\xi^2\frac{d\theta}{d\xi}\right)_1 < \left(-\xi^2\frac{d\theta}{d\xi}\right)_0.$$

Hence it follows that for given K, M cannot exceed a certain definite limit M_0 given by *

$$M_0 = \frac{4^{\frac{3}{2}}K^{\frac{3}{2}}}{(4\pi)^{\frac{1}{2}}G^{\frac{3}{2}}}\left(-\xi^2\frac{d\theta}{d\xi}\right)_0. \tag{23}$$

For any fixed $M < M_0$, (21) determines a definite and unique solution ξ_1, and (22) then determines the radius r_1 of the configuration of mass M subjected to pressure P_1. This solution ξ_1 is independent of P_1. Hence the coefficient $\xi_1\,\theta_1$ in (22) is independent of P_1 and so for a fixed $M < M_0$

$$r_1 \propto P_1^{-\frac{1}{4}}. \tag{24}$$

In the ordinary theory of the complete $n=3$ polytrope with zero external pressure, it is known that the radius r_1 is arbitrary and the equilibrium "neutral." We see, as might be expected, that this result is radically altered by the introduction of the slightest external pressure P_1. Instead of being arbitrary, the radius is now definite. Moreover, as $P_1 \to 0$, by (24) $r_1 \to \infty$, so that *in the limit of zero pressure at the boundary* the radius of the polytropic $n=3$ sphere tends to infinity instead of to an arbitrary value. By (24) the configuration under external boundary pressure obeys a modified form of Boyle's law of the type

$$(\text{volume}) \times (\text{external pressure})^{\frac{3}{4}} = \text{const.}$$

If we suppose ξ_1 eliminated between (21) and (22), both explicitly and as occurring in $\theta_1 = \theta(\xi_1)$, we have a relation of the form

$$r_1 = f(M)P_1^{-\frac{1}{4}}. \tag{25}$$

To examine the behaviour of $f(M)$, consider ξ_1 as a parameter in (21) and (22). Its value can range from $\xi_1 = 0$ to $\xi_1 = \xi_0$. For ξ_1 small, the series solution gives

$$\theta_1 \sim 1 - \frac{\xi_1^2}{6}, \qquad \left(-\xi^2\frac{d\theta}{d\xi}\right)_1 \sim \frac{\xi_1^3}{3},$$

whence

$$M \sim \frac{4^{\frac{3}{2}}K^{\frac{3}{2}}}{(4\pi)^{\frac{1}{2}}G^{\frac{3}{2}}}\left(\frac{\xi_1^3}{3}\right), \qquad r_1 \sim \frac{4^{\frac{1}{2}}K^{\frac{3}{4}}}{(4\pi)^{\frac{1}{2}}G^{\frac{1}{2}}}P_1^{-\frac{1}{4}}\xi_1.$$

Eliminating ξ_1, we have

$$r_1 \sim \frac{K^{\frac{1}{4}}(3M)^{\frac{1}{3}}}{(4\pi)^{\frac{1}{3}}}P_1^{-\frac{1}{4}}, \tag{26}$$

valid provided M is sufficiently small.

* M_0 is, of course, the mass of the *complete* $n=3$ polytrope.

For ξ near ξ_0, we have

$$\left(-\xi^2\frac{d\theta}{d\xi}\right)_1 \sim \left(-\xi^2\frac{d\theta}{d\xi}\right)_0 + \frac{(\xi_1-\xi_0)^4}{4!}6\xi_0{}^2\left(\frac{d\theta}{d\xi}\right)_0^3,$$

so that

$$\frac{M_0-M}{M_0} = \frac{\left(-\xi^2\frac{d\theta}{d\xi}\right)_0 - \left(-\xi^2\frac{d\theta}{d\xi}\right)_1}{\left(-\xi^2\frac{d\theta}{d\xi}\right)_0} \sim \frac{(\xi_1-\xi_0)^4}{4}\left(-\frac{d\theta}{d\xi}\right)_0^2,$$

and

$$\theta_1 \sim (\xi_1-\xi_0)\left(\frac{d\theta}{d\xi}\right)_0$$

$$\sim 2^{\frac{1}{2}}\left(\frac{M_0-M}{M_0}\right)^{\frac{1}{4}}\left(-\frac{d\theta}{d\xi}\right)_0^{\frac{1}{2}}.$$

Hence

$$r_1 \sim \frac{4^{\frac{1}{2}}K^{\frac{3}{4}}}{(4\pi)^{\frac{1}{2}}G^{\frac{1}{2}}}P_1{}^{-\frac{1}{4}}2^{\frac{1}{2}}\left(\frac{M_0-M}{M_0}\right)^{\frac{1}{4}}\xi_0\left(-\frac{d\theta}{d\xi}\right)_0^{\frac{1}{2}}, \tag{27}$$

valid for M sufficiently close to M_0. In particular, for $M=M_0$, $r_1=0$. Thus we have the two results: (i) that for any boundary pressure, a mass equal to that of the "complete" polytrope has zero radius; (ii) for any mass less than that of the complete polytrope, for the limiting case of zero boundary pressure the radius is infinite. The former of these results is in harmony with the "neutral" character of the equilibrium of the complete polytrope, and both results illustrate the "devastating effect" of the imposition of even a small boundary pressure in the case of the "$n=3$" polytrope, as I have pointed out previously.* A mass exceeding M_0 cannot be held in equilibrium by any boundary pressure whatever when $n=3$, for given K. The only configuration is presumably then a point-mass.

Case 2. $n\neq 3$. When $n\neq 3$, we can solve (13) for P_1 in terms of M and ξ_1, and then, substituting this in (14), obtain r_1 in terms of M and ξ_1. The result is to express P_1 and r_1 parametrically in terms of ξ_1. We find

$$P_1 = \left[\frac{(4\pi)^{\frac{1}{2}}G^{\frac{3}{2}}M}{(n+1)^{\frac{3}{2}}}\right]^{-\frac{2(n+1)}{n-3}} K^{\frac{4n}{n-3}}\theta_1{}^{n+1}\left[-\xi^2\frac{d\theta}{d\xi}\right]_1^{\frac{2(n+1)}{n-3}}, \tag{28}$$

$$r_1 = \left[4\pi\left(\frac{G}{(n+1)K}\right)^n M^{n-1}\right]^{\frac{1}{n-3}}\xi_1\left[-\xi^2\frac{d\theta}{d\xi}\right]_1^{-\frac{n-1}{n-3}}. \tag{29}$$

Given M, K and n, a knowledge of P_1 fixes ξ_1 by (28), and this value of ξ_1 inserted in (29) then determines r_1. Elimination of ξ_1 between (28) and (29) yields a relation of the type

$$r_1 = \left[4\pi\left(\frac{G}{(n+1)K}\right)^n M^{n-1}\right]^{\frac{1}{n-3}} f\left(P_1K^{-\frac{4n}{n-3}}\left(\frac{(4\pi)^{\frac{1}{2}}G^{\frac{3}{2}}M}{(n+1)^{\frac{3}{2}}}\right)^{\frac{2(n+1)}{n-3}}\right). \tag{30}$$

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* *M.N.*, **90**, 22, 1929, where similar results were sketched.

We examine the proportionalities corresponding to this equation for ξ_1 small and ξ_1 near ξ_0 respectively.

For ξ_1 small,

$$\theta_1 \sim 1 - \frac{{\xi_1}^2}{6}, \qquad \left(-\xi^2 \frac{d\theta}{d\xi}\right)_1 \sim \frac{{\xi_1}^3}{3},$$

and (28) and (29) give

$$P_1 \propto M^{-\frac{2(n+1)}{n-3}} {\xi_1}^{\frac{6(n+1)}{n-3}}, \tag{31}$$

$$r_1 \propto M^{\frac{n-1}{n-3}} {\xi_1}^{-\frac{2n}{n-3}}. \tag{32}$$

For $\xi_1 \sim \xi_0$, θ_1 small,*

$$P_1 \propto M^{-\frac{2(n+1)}{n-3}} {\theta_1}^{n+1}, \tag{33}$$

$$r_1 \propto M^{\frac{n-1}{n-3}}. \tag{34}$$

Two sub-cases now arise, according as $n < 3$ or $n > 3$.

Case 2a. $n < 3$. For ξ_1 small, P_1 is large and r_1 is small, and eliminating ξ_1 between (31) and (32) we find

$$r_1 \propto M^{\frac{1}{3}} {P_1}^{-\frac{n}{3(n+1)}}. \qquad (r_1 \text{ small}) \tag{35}$$

For $\xi_1 \sim \xi_0$, P_1 is small, r_1 is roughly independent of P_1, and finite and non-zero, and

$$r_1 \propto M^{-\frac{n-1}{3-n}}. \qquad (r_1 \text{ finite}) \tag{36}$$

Case 2b. $n > 3$. For ξ_1 small, P_1 is small and r_1 is large, and as before

$$r_1 \propto M^{\frac{1}{3}} {P_1}^{-\frac{n}{3(n+1)}}. \qquad (r_1 \text{ large}) \tag{37}$$

For $\xi_1 \sim \xi_0$, P_1 is small, r_1 is again roughly independent of P_1, and finite and non-zero, and

$$r_1 \propto M^{\frac{n-1}{n-3}}. \qquad (r_1 \text{ finite}) \tag{38}$$

In case (2*a*) it is readily found that as P_1 increases from zero to infinity, r_1 steadily decreases from the finite limit given by (36) to zero; r_1 never exceeds its value for $P_1 = 0$. In case (2*b*) there is a certain maximum possible value of P_1, say $(P_1)_{\max}$; as P_1 increases from zero to this maximum, r_1 varies continuously from the limit given by (38) to a certain finite value, then, as P_1 decreases again from this maximum to zero, r_1 steadily increases to infinity. For any value of P_1 less than $(P_1)_{\max}$, there are thus when $n > 3$ two possible values of the radius r_1; for P_1 small, one of these values is finite and near the radius of the "complete" n-polytrope, and the other is very large.

It is easily seen in fact by inspection of (28) that when $n > 3$, for given P_1 (28) has two solutions ξ_1. For the function

$$\left[\theta_1 \left(-\xi^2 \frac{d\theta}{d\xi}\right)_1^{\frac{2}{n-3}}\right]^{n+1} \tag{39}$$

* Here ξ_0 denotes the value of ξ for which θ vanishes for the polytrope defined by n. (We assume $n < 5$, so that ξ_0 exists.) The root ξ_0 is, of course, a function of n.

vanishes for ξ_1 small and for θ_1 small, *i.e.* for ξ_1 near ξ_0. Thus in the range $(0, \xi_0)$ it has at least one maximum, and use of the tables of Emden functions * shows that there is just one maximum. Hence there are two solutions ξ_1 for any given P_1 and M_1 and accordingly two distinct radii r_1. At $P_1 = (P_1)_{\max}$, the two solutions r_1 coincide and the radius is slightly indefinite. When $n < 3$, on the other hand, the function (39) is monotonic decreasing, and (28) has only one root.

5. These relations are exhibited graphically in figs. 1, 2, 3, where r_1 is plotted against P_1 on a scale which depends on M. Actually, by (28) and (29), it is sufficient to plot, in the cases (2*a*) and (2*b*),

$$\xi_1\left(-\xi^2\frac{d\theta}{d\xi}\right)_1^{-\frac{n-1}{n-3}} \quad \text{against} \quad \theta_1{}^{n+1}\left(-\xi^2\frac{d\theta}{d\xi}\right)^{\frac{2(n+1)}{n-3}}.$$

In case 1, the relation is simply $r_1 \propto P_1^{-\frac{1}{4}}$.

6. The results are readily interpreted physically. In fig. 1 ($n < 3$, case 2*a*) points near $\xi_1 \sim 0$ correspond to nearly homogeneous configurations of small radius under high external pressure. As the external pressure diminishes, the configuration expands, gaining concentration of matter towards the centre, until for $\xi_1 \sim \xi_0$ we have a nearly complete polytrope with not too pronounced density-concentration towards the centre. For K constant, the volume diminishes as the pressure increases, and accordingly *subject to the "constraint"* † $K = \text{const.}$ the configurations are *stable*. In fig. 2 ($n = 3$, case 1) the value of ξ_1 is determined solely by the value of M; small masses are nearly homogeneous, masses M near $M = M_0$ have nearly the density-concentration of the well-known complete "$n = 3$" polytrope. The volume diminishes as the pressure increases, and subject again to the constraint $K = \text{const.}$ the configurations are *stable* and possess determinate and unique radii. In fig. 3 ($n > 3$, case 2*b*) points near $\xi_1 \sim 0$ correspond to nearly homogeneous configurations of very large radius under small external pressure. As the external pressure increases, the configuration at first contracts, gaining concentration of density towards the centre, until a certain maximum possible external pressure is reached. At this stage the concentration of matter towards the centre is so pronounced, and consequently the internal gravitational attraction so strong, that further increase of pressure would cause complete collapse of the configuration to a point mass. This point, E in fig. 3, is a point of bifurcation where two linear series of equilibrium configurations meet, and consequently here there should be a change of stability. In confirmation of this we notice that whereas along the branch *FE increasing* pressure corresponds to *decreasing* volume, along part of the branch *ED decreasing* pressure corresponds to *decreasing* volume, and thus, on ordinary thermodynamic grounds, *FE* is a locus of *stable* configurations, part of *ED* a locus of *unstable* configurations. At D the configuration is that of a complete stable polytrope under zero external pressure at finite

* Comrie, Miller and Sadler, *Mathematical Tables of the British Association*, vol. 2 (*Emden Functions*), 1932.

† We use the word "constraint" in the technical sense of an imposition of a relation between the parameters defining the configurations of equilibrium.

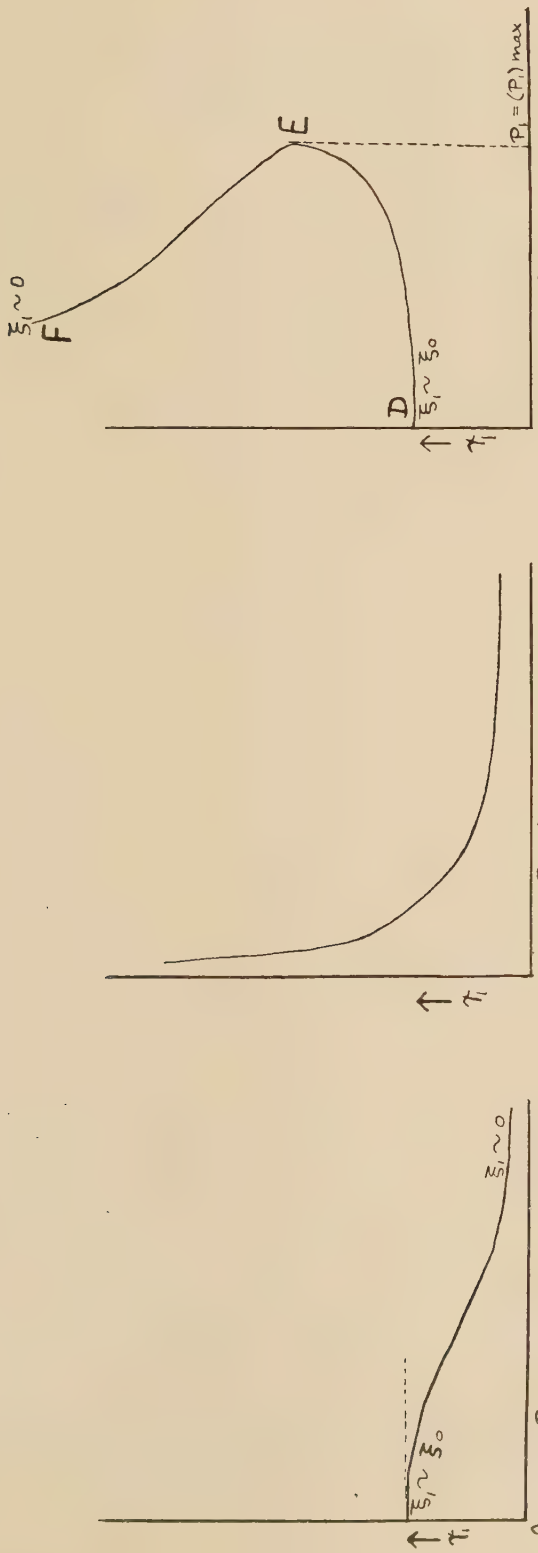

FIG. 1. $n < 3$ (case 2*a*).

FIG. 2. $n = 3$ (case 1).

FIG. 3. $n > 3$ (case 2*b*).

(*Note.*—These sketches are only intended to represent the rough features of the behaviour of r_1 with P_1 as far as ascertained by the approximations in this paper; they are qualitative only. In particular, they do not represent the actual shapes of the curves near $\xi_1 \sim \xi_0$, which depend upon the next terms in approximations (34), (36), (38). Actually the omitted terms give a rapid decrease of r_1 as P_1 increases from zero near $\xi_1 \sim \xi_0$, and, for $n > 3$, r_1 appears to have a minimum between D and E.)

radius. But it is to be particularly noted that for zero or small pressure there exists when $n > 3$ a second solution of large radius.

That fig. 2 has properties intermediate between those of figs. 1 and 3 is evident from inspection: the case $n = 3$ resembles the case $n < 3$ at small radii and the case $n > 3$ at large radii.

7. I have throughout been considering the abstract problem of polytropic configurations of given K and n and have designedly refrained from using the word "star." A real star may for some purposes be regarded as some kind of polytrope or composite polytrope held down to its observed radius by the photospheric pressure.* But neither the polytropic coefficient K nor the photospheric pressure P_1 are given *a priori*—they depend themselves on the structure of the star. The photospheric level is simply a convenient place at which we may regard interior conditions as giving place to the conditions of a stellar atmosphere; and to regard the radius as the level corresponding to the photospheric pressure is one stage better than to regard the radius as the place where T, ρ, P all vanish, as is done in the conventional treatment of polytropes. (For on any model the polytropic character of the equilibrium must break down outside the photosphere.) But the application of our results to real stars with detailed numerical determination of K and P_1 is such a large subject that it is reserved for separate papers. In the meantime we briefly survey our results so far obtained so as to see what promise they hold of matters of cosmogonic interest.

8. In the first place, a real star is not subject to the constraint $K = \text{const.}$ Hence we cannot carry over to real stars our results on stability as they stand. Our results on stability, in particular the remarkable conclusions for the case $n > 3$, refer to a purely formal, abstract problem, just as the stability of rotating figures of equilibrium of liquids has been investigated for constraints which conserve an ellipsoidal boundary; no real rotating mass could be actually subjected to such a constraint. We have indeed found the paradoxical result that it is the configurations of *larger* radius which are stable, some of those of smaller radius unstable, for $n > 3$. I make no suggestion that this result holds good of actual stars. What, however, is of interest and importance is that we have found the possibility of the existence of two distinct radii under given conditions for $n > 3$. Under certain simplifying hypotheses, Jeans has found that $n = 3{\cdot}25$ might be taken to hold good under actual interior conditions,† and his analysis has recently been recapitulated by Chandrasekhar.‡ Similar calculations have been made by Jeans for the case of departures from perfect-gas conditions. Eddington's well-known "standard model" for perfect-gas conditions gives $n = 3$. Without assenting to all the details of these calculations or to their applicability to real stars, we may say generally that the class of polytropes centring round $n = 3$ is of importance. Degenerate stars correspond roughly to $n = \frac{3}{2}$. The extreme sensitivity of the polytrope $n = 3$ has been commented on by Jeans.§ I by

* This is an entirely new aspect of the problem of stellar equilibrium. Its consequences are of considerable importance, and will be pursued in later papers.

† *Astronomy and Cosmogony* (1928), p. 88.

‡ *M.N.*, 95, 684, 1935.

§ *Loc. cit.*, p. 82.

no means assent to his interpretation of the properties of "$n=3$" polytropes, but our result that the value $n=3$ separates polytropes complete or incomplete into classes with very dissimilar properties is not in disharmony with Jeans's conclusions. Should stars prove to be built approximately according to $n=3{\cdot}25$, our result that the incomplete polytropes defined by given boundary pressure can have two distinct radii for $n>3$ is highly suggestive. It is especially notable that for small boundary pressures one of the radii is approximately that of the complete Emden polytrope, whilst the other is very large, for this suggests the possibility of the existence of two classes of stars, of "normal" radius and "very large" radius respectively, corresponding to the observed existence of main sequence stars and giant stars or super-giant stars. I must repeat that at this stage I am putting forward no definite theory to this effect ; I am simply exploring the suggestions of our analysis, reserving detailed investigations till later, when I shall link together the whole of my investigations of stellar structure.

9. In the second place, the existence of two distinct radii for $n>3$ provides for the occurrence of *novæ*. I have on numerous occasions * suggested that the origin of novæ may be looked for in the collapse of a star from a configuration of large radius to one of small radius, with disengagement of a store of gravitational energy and consequent driving-off of the outer portions under radiation pressure. The suggestion originated in the possibility of the passage through a critical luminosity, or in that of a transition between two composite configurations of the same luminosity but unequal radii. In these cases the final state should be a dense star, and difficulties have arisen since not all ex-novæ appear to be identifiable with stars in the white-dwarf stage. We now see that the original suggestion is of possibly wider application. If the suggestion of § 8 above as to giants or super-giants and main-sequence stars is admitted, then it is possible for a transition of one and the same mass to occur, under given boundary pressure, from a large radius to a normal radius, *i.e.* from a giant state to a main-sequence state. In this case the stability of the larger-radius configuration would be of the character of metastability, and the transition could be undergone by any giant star. I shall not here pursue the suggestion further.

10. In the third place, the assumption of a definite radius by an "$n=3$" polytrope under given boundary pressure is of fundamental importance. It has previously been supposed, owing to the "neutral" character of the equilibrium of the *complete* polytrope for $n=3$, that such a configuration could adjust its radius so as to adapt itself to internal conditions ; for example, in Eddington's treatment, the radius adjusts itself to the luminosity, *via* a condition arising from internal opacity. This adjustment is now seen to be no longer permitted in this form ; the adjustment must be to the photospheric pressure, the photospheric pressure itself being determined by the surface gravity, the effective temperature and the atmospheric opacity.†

* *M.N.*, **91**, 29, 1930 ; *Observatory*, **54**, 126 and 140, 1931 ; *Nature* (B.A. Discussion, 1931), 1931 October 24 ; *M.N.*, **92**, 632, 1932 ; *Halley Lecture* ("The White Dwarf Stars"), 1932, p. 11. See also McCrea, *Nature*, 1935 September 28, p. 501.

† *Phil. Trans.*, **228**, A, 456, 1929. Also S. Chandrasekhar, *M.N.*, **92**, 186, 1932.

I write this paper with full knowledge of the developments which these considerations imply, but I do not propose to anticipate them here. It will be clear to the reader, however, that photospheric opacity plays a rôle in determining the radius of a star in terms of its luminosity and mass, as I originally showed by an essentially similar method * in 1929. For the moment the interest lies in the complete and unique determinacy of the radius of the $n \leqslant 3$ polytropes under known external pressure; but the uniqueness is lost as soon as n rises infinitesimally above 3. Since n is a measure of the distribution of internal density, it is again clear that the character of the configurations is a function of the internal density-distribution which changes decisively at a certain stage of density-concentration towards the centre. These results depend simply on the degree of compressibility of the material—we have nowhere introduced, as yet, temperatures or assumed any equation of state,† or separated gas-pressure and radiation-pressure.

11. In the fourth place we see the emergence at $n=3$ of a critical mass M_0, such that, when $n=3$, configurations under given external pressure, for given K, only exist when $M < M_0$. Until we discuss the values of K and P_1 we cannot estimate M_0. But investigators have long looked for a condition setting an upper limit to the mass of a system of stellar character, and the emergence of an M_0 is in harmony with Eddington's conjectures on the subject. The proper discussion of the standard model with the proper conditions for radiative equilibrium at the boundary, which I have carried out in unpublished investigations, also seems to set an upper limit to M. Our present investigation is merely an indicator of possibilities.

12. Lastly, attention may be directed to the emergence of a maximum pressure that can be sustained by a polytrope for $n > 3$ for given K and M. The evaluation of this maximum pressure will be undertaken later for stars of given mass and effective temperature. I may anticipate my detailed results by stating that this maximum pressure is of the order of, but somewhat greater than, the photospheric pressure for a star like the Sun. For stars with a high value of radiation pressure near the boundary very interesting possibilities arise. For the present the significant point is that at $P_1 = (P_1)_{\max}$ the radius is very sensitive to small variations in P_1, and hence the radius is here not well defined. These configurations have a radius intermediate between the large-radius configurations and the small-radius configurations which correspond to smaller values of P_1. Now Cepheid variation appears to occur for radii intermediate between those of the giant M-type stars and the B-type stars, namely, chiefly at types F and G, and Cepheid variation is associated with periodic pulsations in radius. These would be apt to be

* *M.N.*, 90, 17, 1929. As this paper has not been generally accepted, I take this opportunity of re-affirming the general nature of the conclusions it arrived at, more particularly the conclusions stated in its opening paragraph. It is now possible, however, to present the argument in a much more convincing way, and much more simply, from the starting-point of the present paper.

† The polytropic relation $P = K\rho^{1+\frac{1}{n}}$ is not an equation of state, as it does not involve the temperature; it is merely a statement of the degree of compressibility

originated in configurations where the radius was normally (in equilibrium) slightly indeterminate. These considerations suggest that Cepheid variation may be associated with the phenomenon of the occurrence of a maximum possible boundary pressure for $n > 3$—a hypothesis similar to one I have made previously * in a slightly different connection, where indeterminacy of radius accompanied a maximum possible luminosity. In the latter case, according to Fairclough's calculations,† the calculated radii came out too small; on the present variant of the hypothesis the calculated radii come out much larger, if not too large. Again, in this paper I content myself with indications.

13. *Summary.*—The radii of polytropic distributions under given external pressure are investigated in a formal manner with a view to subsequent applications to actual stars. It is shown that for $n \leqslant 3$ there is a unique radius for every value of the external-pressure, but for $n > 3$ there is a maximum permissible external-pressure; for any external-pressure less than this there are two distinct values of the radius. For $n = 3$ and given K there is a maximum possible mass. Cosmogonic applications are briefly considered.

POLYTROPIC EQUILIBRIUM. II. THE PRESSURE IN THE PHOTOSPHERE OF A STAR.

E. A. Milne.

1. The object of this and the following paper was originally to make numerical application of the formulæ contained in the foregoing paper to actual stars. The reader will find that the centre of interest continually changes as the papers advance, as it did during the research itself, and I shall not anticipate the results.

2. When the degree of compressibility of the matter composing a star is specified by the polytropic relation of Emden,

$$P = K\rho^{1+\frac{1}{n}}, \tag{1}$$

the characteristics of the configuration depend on n and K jointly, in general. The index n, roughly speaking, determines the degree of concentration of the material of the star towards the centre. For a complete polytrope, the ratio of central density to mean density depends only on n, and is independent of K. For an incomplete polytrope, as shown by formula (19) of the preceding paper, this ratio again practically depends only on n when ξ_1, the boundary value of the Emden variable ξ, is approximately ξ_0, the first zero of the associated Emden function. We shall assume that some general discussion of the physics of the interior of the star has settled either the general nature of the compressibility, or the general nature of the degree

* *M.N.*, **92**, 634, 1932. See also *Observatory*, **58**, 175, 1935.

† *M.N.*, **95**, 585, 1935.

of concentration of material towards the centre, and so settled the approximate value of n ruling through the interior, or some mean value for n; or alternatively that it has settled n directly. I want to make it clear that some preliminary interior considerations are supposed to have led to a knowledge of n, so that the analysis which follows, in so far as it involves n, does really pre-suppose some knowledge of, or hypotheses concerning, the conditions in the interior. The analysis will be indissolubly bound up with the equilibrium of the interior. It is not necessarily supposed that this value of n holds good exactly at all points in the interior, in particular it is not necessarily supposed that this value of n holds good for the compressibility of the layers immediately beneath the photosphere.

The absolute values of the various variables descriptive of the interior depend on the value assigned to K. Let ρ_1 be the actual density at optical depth $\tau = \frac{2}{3}$, a level we shall call the photospheric level, and let P_1 be the total pressure at the same depth. I propose to fix K by defining it as the number satisfying the relation

$$P_1 = K\rho_1^{1+\frac{1}{n}}. \tag{2}$$

This choice of K may cause the reader difficulties. It is sometimes stated that a formula of the type (1), valid for the far interior, cannot be expected to hold good in the exterior layers of the star. The origin of this view appears to lie partly in the circumstance that in attempting to derive a formula of type (1) from assumed stellar conditions, authors sometimes use inverse powers of the temperature, and under the usual treatment which considers the boundary as given by $T = 0$, the formula becomes invalid at the boundary. But we are cutting off our configuration at the photosphere —we are not proposing at all to apply (1) outside the photosphere. Secondly, on older views, as the ionisation must change markedly as the outer layers are approached from the inside, at least for elements of higher atomic number, the mean molecular weight was supposed to change considerably as the photosphere was approached, and so the compressibility. But if stars are largely composed of hydrogen, for a star like the Sun the mean molecular weight can only vary slightly, in the neighbourhood of unity in the photospheric layers, to at most double this in the interior. Thus there is no longer any reason to expect marked changes of polytropic compressibility in the outer layers.

But even if they did occur they are irrelevant to our present problem. If it is objected that the choice (2) for K cannot reproduce accurately the pressures and densities in the interior, I reply that we know nothing *directly* about the interior, and that the value (2) for K, combined with the adopted n, will produce definite pressures and densities in the interior which are compatible with mechanical equilibrium and which will give the desired ratio of central to mean density; for the latter depends (in most cases) almost solely on n alone. Further, the choice (2) for K ensures continuity of values of P and ρ between interior and atmosphere, at the photospheric level. If we demand this continuity, as we must, we have no alternative but to fix K by (2), if we are approximating to reality by taking a fixed mean value for n.

If the reader is still dissatisfied, I would ask him to consider n and K as determined by (2) together with the central relation $P_c = K\rho_c^{1+\frac{1}{n}}$. For any given values of P_1, ρ_1, P_c, ρ_c these two equations can be solved for n and K, and the polytropic equation (1) is then to be regarded as an interpolation formula giving a continuous and monotonic increase of P with monotonic increase of ρ from photosphere to centre, continuous with the actual values of P and ρ at photosphere and centre. Actually, as we do not know P_c and ρ_c, it is better to adopt a value for n on other grounds, and then find K from (2).

The main point is that (1), combined with a choice of K given by (2), provides a definite stellar model with correct boundary values. Since the boundary values are all we can observe, or all we could measure by an actual visit to the accessible part of a star, it is more important to have the boundary values correct than to rely on hypotheses about the interior. We do indeed rely on a physics of the interior to provide a value for n, in the first instance; * but I am not choosing n to fit the layers immediately beneath the photosphere—I do not assume that their compressibility is governed in detail by (1). It is not to be supposed that we are *extrapolating* (1) to the photosphere when we write down (2); relation (2) *defines* the model we are choosing to consider. Its utility will be checked later by actual calculation of the photospheric pressure P_1 and other relations it leads to. We are setting up a model correct at the photosphere and we propose to inquire where this takes us to.†

3. Relation (1) and definition (2) being adopted, we can now use formulæ (16) and (17) of the preceding paper. These state that the total mass M inside the photosphere is held down under photospheric or "boundary" pressure P_1 to a radius r_1 defined by the relations

$$M = \frac{(n+1)^{\frac{3}{2}}}{(4\pi)^{\frac{1}{2}}G^{\frac{3}{2}}} \frac{P_1^{\frac{3}{2}}}{\rho_1^2} \theta_1^{\frac{n-3}{2}} \left(-\xi^2 \frac{d\theta}{d\xi}\right)_1, \tag{3}$$

$$r_1 = \frac{(n+1)^{\frac{1}{2}}}{(4\pi)^{\frac{1}{2}}G^{\frac{1}{2}}} \frac{P_1^{\frac{1}{2}}}{\rho_1} \xi_1 \theta_1^{\frac{n-1}{2}}, \tag{4}$$

where ξ_1 may be considered simply as a parameter and θ_1 or $\theta(\xi_1)$ is the value at $\xi = \xi_1$ of Emden's solution ‡ of the polytropic equation

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2 \frac{d\theta}{d\xi}\right) = -\theta^n.$$

The suffix 1 denotes evaluation at $\xi = \xi_1$.

* The value of n will be seen later to be approximately determinable from surface data.

† It is particularly to be noted that I do not assume the photospheres of different stars to be at "homologous" points. They prove later not to be. Each star is considered in the first instance on its own merits.

‡ The whole of the analysis of this paper applies also for non-Emden solutions, the use of which would be appropriate when n and K change discontinuously in the interior, as in composite polytropes.

Given M, P_1, ρ_1 and a table of values of $\theta(\xi)$ and its differential coefficient, equation (3) can be solved for ξ_1 in general, and insertion of this value ξ_1 in (4) determines the radius r_1. Equations (3) and (4) depend on the mechanical equilibrium of the interior and on the polytropic relation (1), and on nothing else. They are *interior* relations fixing simply the external size of a mass M held down by pressure P_1 when its compressibility is governed by (1). They involve only ordinary hydrostatics and the law of gravitation.

4. So far we have not assumed any equation of state—temperature has not been involved. If we could measure ρ_1 directly, the introduction of temperature at this point would be superfluous. We could in principle eliminate the parameter ξ_1 between (3) and (4), and so obtain a relation between M, r_1, ρ_1 and P_1. We could calculate P_1 from this, and compare it with the value derived from a study of stellar atmospheres. If it was discordant, we could try again with a different value of n, adjusting n until P_1 came right. We should thus find, from surface data, a definite piece of information about the interior, namely, its relative density-distribution as specified by n.

Actually we cannot observe ρ_1 directly. We observe T_e, the effective temperature of the star; or from spectroscopic observations we determine the actual temperature of the stellar atmosphere. Now, in the outer layers, the temperature T, on the verified assumption of radiative equilibrium *in the outer layers*, is governed by the equation

$$T^4 = \tfrac{1}{2} T_e^4 (1 + \tfrac{3}{2}\tau). \tag{5}$$

We have defined P_1 and ρ_1 as the values of the variables P and ρ at optical depth $\tau = \frac{2}{3}$. Hence by (5) if T_1 is the temperature at this depth,

$$T_1^4 = T_e^4. \tag{6}$$

Accordingly $T_1 = T_e$, and so given T_e, T_1 is fixed.

We now use the fact that the photospheric layers are known to be gaseous. Let R be the gas-constant, μ the mean molecular weight *at the photosphere*; μ has been determined spectroscopically from measures of the composition and ionisation of stellar atmospheres by Russell, Unsöld and others and will be supposed known. Let p_1 be the gas pressure at the photosphere. Then

$$p_1 = (R/\mu)\rho_1 T_1. \tag{7}$$

Finally, let β_1 be the ratio of gas pressure to total pressure at the photosphere, so that

$$P_1 = p_1/\beta_1. \tag{8}$$

5. In (3) substitute for ρ_1 from (7) and for P_1 from (8) and solve the resulting equation for p_1. We obtain

$$p_1 = \frac{(R/\mu)^4 T_1^4}{4\pi G^3 M^2 \beta_1^3} (n+1)^3 \theta_1^{n-3} \left(-\xi^2 \frac{d\theta}{d\xi} \right)_1^2. \tag{9}$$

Make the same substitutions in (4) and eliminate * p_1 by means of (9). We obtain

$$r_1 = \frac{\beta_1 GM}{(R/\mu)T_1}\,\frac{1}{n+1}\,\frac{\theta_1}{\left(-\xi\frac{d\theta}{d\xi}\right)_1}. \qquad (10)$$

Formulæ (9) and (10) are the fundamental formulæ of the present paper. We shall apply them in several distinct ways.

6. We first examine the order of magnitude of p_1 implied by (9) for the Sun.

For the Sun, and indeed for all dwarf stars, radiation pressure in the photospheric layers is negligible † compared with gas pressure. Hence in (9) and (10) we may put $\beta_1 \sim 1$.

When $n=3$, the last two factors in (9) reduce to $(-\xi^2 d\theta/d\xi)_1^2$. This certainly cannot exceed $(-\xi^2 d\theta/d\xi)_0^2$, where the suffix 0 means evaluation at $\xi=\xi_0$, the first zero of the Emden function for $n=3$, *i.e.* it cannot exceed $(2{\cdot}01823\ \ldots)^2$. Moreover, by (10), θ_1 cannot be zero, so that ξ_1 must be less than ξ_0. On the other hand, again from (10), θ_1 must be very small for the Sun; for the observed r_1 for the Sun is $6{\cdot}95\times10^{10}$ cms., whilst taking $M=M\odot=1{\cdot}985\times10^{33}$ grams, $(T_1)\odot=5740°$, $\mu=\mu_H=1$, $G=6{\cdot}66\times10^{-8}$, $R=k/m_H=0{\cdot}826\times10^{8}$, the value of $\beta_1 GM/(R/\mu)T_1$ is about $2{\cdot}79\times10^{14}$ cms. Hence in (10) θ_1 must be a small multiplier, ξ_1 must be very close to the zero ξ_0, and $(-\xi^2 d\theta/d\xi)_1$ must be very close to $(-\xi^2 d\theta/d\xi)_0^2=(2{\cdot}01823\ \ldots)^2$. Hence *if* $n=3$, formula (9) would give for the pressure p_1 in the Sun's photosphere the value

$$p_1 = \frac{(0{\cdot}826\times10^{8})^4\times(5740)^4}{4\times3{\cdot}1416\times(6{\cdot}66\times10^{-8})^3\times(1{\cdot}985\times10^{33})^2}\times(3+1)^3\times(2{\cdot}018\ \ldots)^2$$
$$= 221\times(2{\cdot}018)^2 \text{ dynes cm.}^{-2}. \qquad (11)$$

In his table of pressures ‡ in the Sun's photospheric layers, Chandrasekhar gives for $\tau=0{\cdot}5688$, $p=164{\cdot}6$ dynes cm.$^{-2}$, for $\tau=1{\cdot}0410$, $p=260{\cdot}4$ dynes cm.$^{-2}$, whence by linear interpolation at $\tau=\frac{2}{3}$ we get $p_1=184{\cdot}4$ dynes cm.$^{-2}$. It is clear that even for $n=3$ exactly, (9) hits off the correct order of magnitude for the photospheric pressure in the Sun.

If $n\neq3$, we have to retain the factor ${\theta_1}^{n-3}$ in (9). By taking n very slightly higher than 3 and choosing θ_1 so that (10) agrees with the observed radius r_1, the estimate (11) becomes approximately multiplied by ${\theta_1}^{n-3}$, which is less than unity but only a little less than unity for n very slightly exceeding 3. Hence by an appropriate choice of n very slightly greater than 3 we can get complete agreement between the value of p_1 given by (9) and Chandrasekhar's estimate.

* The elimination and substitution are effected in one stroke by using formula (18) of the preceding paper, which was recorded for this purpose.

† See, *e.g.*, Chandrasekhar, *M.N.*, **92**, 186, 1932, Table I.

‡ *Loc. cit.*, Table I. Chandrasekhar adopts a *mean* atomic weight of 1·5 for calculating *opacity*, and one of 0·75 for calculating *heights*. A value of μ in (9) slightly exceeding 1 would improve our agreement.

7. But this is an astonishing result. Chandrasekhar's method of estimating p_1, an adaptation of the method given by me in my Bakerian lecture,* involves essentially a knowledge of the numerical values of the absorption coefficient in the Sun's photospheric layers, and depends on the hydrostatic equilibrium of the layers *above* the photospheric level; it is, in fact, one of the best recognised results in astrophysics that the pressure at the photospheric level, the mean depth to which we see into the star, depends essentially on the opacity of the layers above this depth. On the other hand, (9) depends on the hydrostatic equilibrium of the whole star interior to the photosphere. No estimate of photospheric opacity is required in applying (9), the only solar data employed being the values of M and T_1. The inference that θ_1 is small and ξ_1 consequently near ξ_0 does indeed depend on a knowledge of the *order of magnitude* of r_1, namely, that it is small compared with $GM/(R/\mu)T_1$, but no exact numerical value of r_1 has been employed in arriving at the estimate (11). In fact, if we had assumed on other grounds that the Sun is an almost complete $n=3$ polytrope we should have been led directly to (11) without any observational knowledge of $(r_1)\odot$ whatever. Estimate (11) is evidently an upper limit if $n \geqslant 3$, and we have seen that the possibility that n may slightly exceed 3 permits the small reduction of (11) to agreement with Chandrasekhar's estimate.

8. First of all, this agreement confirms two results of other writers. It confirms the view that the Sun's photospheric layers consist almost wholly of monatomic hydrogen; for we have adopted $\mu=\mu_H=1$, and μ occurs to the inverse fourth power in (9), so that any appreciably higher value for μ would throw out the estimate considerably. And it confirms Eddington's view that the Sun is built very nearly according to a polytrope $n=3$; for any appreciably higher value than 3 for n would multiply the estimate (12) by the then small factor $\theta_1{}^{n-3}$ (besides altering the value of $(-\xi^2 d\theta/d\xi)_1{}^2$ to that for the corresponding n-polytrope for the value ξ_1 inferred from the θ_1 derived from (10)). This possibility of thus fixing n very closely from surface data is one of the most interesting results of the present investigation, to which we shall return later.

But we have not made any assumptions about the interior save that its compressibility and resulting density-distribution nearly correspond to $n=3$. In particular, we have not assumed radiative equilibrium in the interior, we have not assumed obedience to the gas-laws, and we have not made any assumption about the mean molecular weight in the interior or the relative importance of radiation pressure in the interior. For all we are concerned, the Sun may consist partly of non-perfect gas, or may contain a small degenerate core, or may be partly in convective equilibrium; † we have made no assumption about the distribution of sources of energy in the interior, or about the opacity of the interior. It is true that if we

* *Phil. Trans.*, **228**, A, 421, 1929. This method goes back to researches by Woltjer, *B.A.N.*, **2**, 171, No. 66, 1924; Jeans, *M.N.*, **85**, 199, 1925; and myself, *M.N.*, **85**, 784, 1925, and *M.N.*, **90**, 33, 1929.

† *Cf.* Biermann, *A.N.*, **257**, Nos. 6161–62, 1935. Also, for layers *beneath* the photosphere, *cf.* Unsöld, *Zeits. für Astrophys.*, **1**, 138, 1930; **2**, 209, 1931.

assume that the Sun is composed of perfect gas in radiative equilibrium with uniform source-distribution and constant opacity (or alternatively constant "$k\eta$"), then, as shown by Eddington, we have $n=3$; but the argument cannot be reversed.

9. But secondly, and of equal theoretical importance, we have the fact that (9) yields a correct estimate of photospheric pressure without any knowledge of the photospheric opacity. As already stated, the photospheric level is the mean depth to which we see into the star; the more opaque the photospheric layers the smaller the pressure at the depth down to which we can see, and the smaller is the pressure at optical depth $\tau=\frac{2}{3}$. Yet in applying (9) we have used no estimate of photospheric opacity; we have used only values of M and T_1. It is therefore evident that the Sun must have adjusted its photospheric temperature T_1 so as to fit its photospheric opacity.* The physical reason for this adjustment is obvious. The photospheric pressure is not only the depth down to which we see; it is also the pressure which holds down the compressible mass M to its observed radius r_1. Working inwards through the layers external to $\tau=\frac{2}{3}$ we can calculate the pressure at $\tau=\frac{2}{3}$ from the formula for the general coefficient of opacity in the photospheric layers; working outwards from the centre of the star, we can calculate the pressure at the same level from the compressibility and equilibrium of the interior, taking into account the order of magnitude of the observed radius. Since the pressure must be continuous, the two values of the pressure must agree. This condition has not hitherto been taken into account in theoretical astrophysics. I am putting forward no new theory; I am simply fitting together two accepted domains of theoretical investigation. I am replacing the hitherto adopted but unacceptable boundary conditions, $P=0$, $\rho=0$, $T=0$, by the actual boundary conditions, $P=P_1$, $\rho=\rho_1$, $T=T_1$.

In Part III of this series of papers I combine theoretical calculations of p_1 from the equilibrium of the external layers with the calculation of p_1 from (9). This leads to the relation between the luminosity, mass and effective temperature of a star in terms of the photospheric opacity, or some coefficient which determines this opacity. The formula determines how the effective or photospheric temperature must adjust itself to suit the independent parameters M, L and photospheric opacity coefficient; it will express the condition that the star must adjust itself so as to be capable of radiating to space the energy generated in the interior. Such a wide domain of thought opens up at this point, however, that in the present paper I confine myself to the further consequences of (9) and (10) when p_1 is taken as known, either by spectroscopic determination or from the calculations of others.

10. Let us apply (9) and (10) to Capella (brighter component), ($M=4{\cdot}18\odot$, $T_e=5300°$, $r_1=0{\cdot}9\times10^{12}$ cms.).† The value of $\beta_1 GM/(R/\mu)T_1$ is $\beta_1\times1{\cdot}26\times10^{15}$ cms., as compared with the observed radius of $0{\cdot}9\times10^{12}$ cms.

* The adjustment is a mutual one, since the photospheric opacity itself depends on T_1.

† B. Strömgren, *Zeits. für Astrophys.*, **7**, 232, 1933, Table 4.

I shall show how to calculate β_1 presently; it may be somewhat less than unity, but as far as orders of magnitude are concerned it may be again taken as unity, and so again, by (10), θ_1 must be small. The actual root ξ_1 required by (10) will not be the same as for the Sun, so that the photospheres will not be at strictly homologous points. Nevertheless, ξ_1 will be close to ξ_0, and we may proceed as before. We get for $n=3$,

$$p_1 = 9{\cdot}2 \times (2{\cdot}018\ \ldots)^2/\beta_1^3. \qquad (12)$$

Considering the reduced value of surface gravity for Capella as compared with the Sun, this is a very reasonable value for p_1. According to ordinary stellar atmosphere theory, the pressure in the photosphere for stars of approximately the same effective temperature should vary roughly as $g^{\frac{1}{2}}$, apart from a complication due to radiation pressure; for comparison of Capella and the Sun, the ratio is about 0·158, and allowing for some influence of β_1^{-3} in (12), estimate (12) based on interior theory for Capella is in harmony with $p_1 = 184$ for the Sun from the point of view of ordinary stellar atmosphere theory. The factor θ_1^{n-3}, omitted in this estimate, may be expected to be nearer unity than for the Sun.

11. It is now clear that formula (9) may be expected to give the same order of magnitude for the photospheric pressure p_1 in all stars, broadly speaking. For M and T_1 are of the same order of magnitude for the majority of stars. This is a well-known conclusion from the study of stellar spectra in relation to ionisation theory, and it can of course be reached *via* a study of photospheric opacity. Here it follows from interior considerations, if n is 3 or only slightly higher. Stars possessing exceptional values of M or T_1, or exceptional values of n, may have exceptional photospheric pressures, in harmony with the exceptional spectra of such stars. For the white dwarf stars, for which $n=\frac{3}{2}$ approximately, in the interior, the theory may be expected to apply less easily, owing to the differing characters of the interior and gaseous fringe, but here a non-Emden solution for $n=3$ would be appropriate in (9) and (10), to a second approximation; even to a first approximation, for a crude mean value of n lying between 3 and $\frac{3}{2}$, (9) gives θ_1 raised to a negative power, and the value of p_1 will be increased, again in accordance with the stellar atmosphere point of view, which would demand higher values for p_1 in accordance with increased surface gravity. But composite polytropes, for which n is essentially discontinuous inside the star, will require a special treatment not here given.

Had formula (9) been known before any spectroscopic theory was applied to stellar atmospheres, before the days of the Saha theory, it could have been correctly applied to infer that the photospheric pressure in the majority of the stars is of the order of a ten-thousandth or a few ten-thousandths of an atmosphere—a result only, in fact, known after Russell and Stewart's early work * and Fowler and Milne's work † on maxima of absorption lines. In the early days of astrophysics, when pressure shifts of spectral lines were being investigated, it was supposed that the pressure in the Sun's reversing

* *Astrophys. Journ.*, 59, 197, 1924.
† *M.N.*, 93, 403, 1923.

layer was of the order of one atmosphere or a few atmospheres.* This erroneous conclusion could have been avoided by a simple application of (9), that is by a recognition of the fact that the photospheric pressure is also the pressure required to hold down the compressible interior to its observed radius. Some knowledge of the interior would indeed have been required, namely, that n is approximately 3.†

Formula (9) *in this form* is new to astrophysics and it appears to be of some importance. We consider some further consequences.

12. Let us suppose that by spectroscopic methods or otherwise the photospheric pressure p_1 has been determined for a number of stars of observed M, r_1 and T_1. (What is called the method of spectroscopic parallaxes, properly considered, is the determination of the pressures in the stellar atmospheres of stars of given T_1.) Then $\beta_1 = p_1/(p_1 + \frac{1}{3}aT_1^4)$ is known. Put

$$V = (n+1)\left(-\frac{\xi}{\theta}\frac{d\theta}{d\xi}\right), \qquad W = (n+1)^3\theta^{n-3}\left(-\xi^2\frac{d\theta}{d\xi}\right)^2. \tag{13}$$

Let a diagram be prepared in which, for a range of values of n, $\log(1/V)$ is plotted against $\log W$, for Emden functions θ. This can be carried out by use of Comrie's "Tables of Emden Functions"‡ where V is in fact tabulated. The result will be a family of curves, each labelled with a value of n. On this diagram, for each observed star, plot the point of co-ordinates

$$y = \log\left[\frac{(R/\mu)Tr_1}{\beta_1 GM}\right], \qquad x = \log\left[\frac{4\pi G^3M^2\beta_1{}^3p_1}{(R/\mu)^4T_1{}^4}\right].$$

Then the n-curve which passes through the plotted point will determine the polytropic index appropriate to the star in question, by (9) and (10). In this way we could find out one fact about the interior (namely, the mean value of n) from the observable external characteristics of the star. How practicable is this method I cannot say, but I hope to attempt it later. The main point is that we have devised in principle a method of determining n without hypotheses about the interior.

13. If n can be found, elimination of ξ_1 between (9) and (10) gives in principle a relation between p_1, M, T_1, r_1 from which any one variable could be determined in terms of data for the other three. It is clear, however, that it is very difficult to determine r_1 accurately when n is near 3, if $p_1M^2/T_1{}^4$ is such that ξ_1 is near ξ_0; for then θ_1 is very small, and (9) gives a bad determination of θ_1.

The function W, for $n > 3$, is zero for $\xi = 0$ and zero for $\xi = \xi_0$, having a single maximum in between. For $n = 3{\cdot}5$, from Comrie's Tables the maximum of W is about 143, and occurs for $\xi_1 = 3{\cdot}9$, $\theta_1 = 0{\cdot}279$; for $n = 4$,

* Jewell, Mohler and Humphreys, *Astrophys. Journ.*, **3**, 198, 1896; Fabry and Buisson, *Astrophys. Journ.*, **31**, 113, 1910; *C.R.*, **148**, 688, 1909.

† As conjectured originally in 1913 by T. Bialobjeski, *Bull. Internat. Acad. Sci. Cracovie*, p. 264, 1913. See G. Tiercy, *L'Équilibre Radiatif dans les Étoiles* (Paris, 1935), p. 47.

‡ *Math. Tables of the Brit. Assoc.*, vol. 2, 1932.

the maximum of W is about 103 and occurs for $\xi_1 = 3{\cdot}5$, $\theta_1 = 0{\cdot}373$; for $n = 3$, of course, the maximum occurs for $\xi_1 = \xi_0 = 6{\cdot}8968\ldots$, and is about 261. For any value of

$$\frac{4\pi G^3 M^2 \beta_1{}^3 p_1}{(R/\mu)^4 T_1{}^4}$$

less than this maximum (for given n) there are two solutions ξ_1, of which, for $M^2 p_1/T_1{}^4$ small, one occurs near $\xi_1 = \xi_0$ and the other near $\xi_1 = 0$; of the corresponding radii one is "normal," the other very large. Values of the above parameter exceeding the maximum cannot exist. For the large radius solution r_1 would be somewhat smaller than $GM/(R/\mu)T_1$, since $1/V$ mostly is less than unity (for $n = 3{\cdot}5$, $\xi_1 = 3{\cdot}9$, V_1 is 7·104; for $n = 4$, $\xi_1 = 3{\cdot}5$, V_1 is 5·682), and there is the factor β_1. But $GM/(R/\mu)T_1$ for most stars seems of too large an order for this second solution r_1 to give the radii even of giant stars. It is therefore not possible at present to say that this second solution is ever realised for actual stars. The possibility remains that it may be realised for super-giants.*

The sensitive factors in (9) and (10) are $\theta_1{}^{n-3}$ and θ_1 respectively. For θ_1 small, the remaining Emden factors are insensitive. We may therefore eliminate θ_1 alone, and get the proportionality

$$\frac{M^2 \beta_1{}^3 p_1}{T_1{}^4} \propto \left[\frac{r_1 T_1}{\beta_1 M}\right]^{n-3}$$

or

$$p_1 \propto \beta_1{}^{-n} M^{-(n-1)} r_1{}^{n-3} T_1{}^{n+1}, \tag{14}$$

where the factor of proportionality is not quite an absolute constant. For given r_1, (10) gives a unique solution ξ_1, since V is monotonic, and hence there is just one solution p_1 for given r_1. We shall employ (14) in the next paper.

14. I have postponed till now a transformation of (9) which may have already occurred to the reader. I have done so because this transformation, though putting (9) in an, at first sight, more familiar form, obscures its physical meaning. The temperature T_1 occurs to the fourth power in (9). This is not due to any introduction of radiation pressure—there has been no introduction, in deriving (9), of the radiation constant a; the occurrence of the fourth power is an algebraic accident,† of which, however, we can make use. Multiply numerator and denominator of the right-hand side of (9) by $\frac{1}{3}a$. Then using

$$\beta_1 = \frac{p_1}{p_1 + \frac{1}{3}aT_1{}^4}, \qquad \frac{\beta_1}{1-\beta_1} = \frac{p_1}{\frac{1}{3}aT_1{}^4}$$

we get

$$\frac{\beta_1{}^4}{1-\beta_1} = \frac{(R/\mu)^4}{4\pi G^3 M^2 (\frac{1}{3}a)}(n+1)^3 \theta_1{}^{n-3}\left(-\xi^2\frac{d\theta}{d\xi}\right)_1^2. \tag{15}$$

* Composite polytropes will, however, require separate consideration; their radii may come out smaller (if previous calculations are here any guide), and accordingly the second solution may be realised in nature in these cases.

† I have drawn attention to this "accident" previously, *M.N.*, **90**, 29, 1929.

In this form it determines from interior considerations the photospheric value β_1 of the ratio of gas pressure to total pressure in terms of the mass M, photospheric molecular weight μ and a parameter ξ_1, which in turn is given by (10) in terms of r_1, M, T_1, μ; all for given n.

Equation (15) in this form is not new. I gave a more general form in a previous paper,* in the form

$$\frac{\beta_1^4}{1-\beta_1}=\frac{(R/\mu)^4}{4\pi G^3M^2(\frac{1}{3}a)}\cdot\frac{4^2}{C}, \qquad C^{-1}=\phi_1^{-1}\left(-x_1^2\frac{d\theta}{dx}\right)_1^2. \tag{16}$$

I re-derived it in a second paper,† where I showed that it reduced to a relation obtained by Eddington for the special case $n=3$ when all his conditions are satisfied. And I derived the $n=3$ case, when L and M are independent variables and the conditions of the standard model are satisfied, in another paper,‡ in the form

$$\frac{\beta^4}{1-\beta}=\frac{(R/\mu)^4}{4\pi G^3M^2(\frac{1}{3}a)}\frac{4^3}{C}, \qquad C^{-1}=\left(-\xi^2\frac{d\theta}{d\xi}\right)_0^2 \tag{17}$$

where I called C the discriminant. (Relation (15) in fact reduces to Eddington's relation (17) for $n=3$ on taking $\xi_1=\xi_0$.) But in these various derivations the complementary relation (10) was not given, and this may account for the misunderstandings to which my work of 1929 gave rise. Relation (15) is, in fact, meaningless without (10), for ξ_1 is unknown, and we need (10) if only to show that θ_1 is small and so ξ_1 near ξ_0. (Of course we cannot put ξ_1 exactly equal to ξ_0 without reducing r_1 to zero.) Relation (15), mathematically equivalent to the above physically significant relation (9), is the polytropic generalisation of (16), and the generalisation of Eddington's quartic equation to cases where $n\neq 3$.

But our derivation of (15) leads to a different interpretation from Eddington's. When β, the ratio of gas pressure to total pressure, is not constant, such a relation provides, not a mean value of β through the interior, but the boundary value, *i.e.* the value at the photosphere. Eddington § and Russell ‖ criticised my relation (16) as being a purely photospheric relation. That this is not so is clear from the present derivation, where (15) only has a meaning in conjunction with (10), and (10) contains the absolute value of the radius r_1; a purely photospheric relation could contain M and r_1 only in the form M/r_1^2. Moreover, n in (9) and (10) does not refer to the photospheric layers, but to the star as a whole. If indeed (15) were a purely photospheric relation, so would Eddington's quartic equation be one. It must be remembered that (17) applies to a different context from (15); in deriving (17) the boundary condition adopted was $T=0$, $P=0$, L and M were independent variables, and (17) determines not β_1 (which is already there, known in terms of L, M and κ_1), but the appropriate singularity-possessing solution of Emden's $n=3$ equation.

* *M.N.*, **90**, 29, 1929, equations (24) and (25).
† *M.N.*, **90**, 687, 1930, equation (19″).
‡ *M.N.*, **91**, 19, 1930, equations (31) and (32).
§ *M.N.*, **90**, 286, 1930.
‖ *M.N.*, **91**, 956–958, 1931.

In all cases the value of C depends on the complete structure of the star, and is found only by solving a differential equation expressing the equilibrium of the interior under its actual compressibility. But it is extremely distasteful to me to refer to these old controversies, and I pass on.

15. Equation (15) can be conveniently used to determine β_1 for any star of given M, r_1 and T_1, assuming it is built according to an n-polytrope; the photospheric pressure p_1 then follows from $p_1 = \frac{1}{3}aT_1^4\beta_1/(1-\beta_1)$ without any knowledge of photospheric opacity. The method is one of successive approximation. I will illustrate it for Capella (A), assuming (again for the sake of illustration) $n = 3{\cdot}5$. For the observed mass M the value of $(R/\mu)^4/4\pi G^3M^2(\frac{1}{3}a)$ is 0·0707. Hence

$$\frac{\beta_1^4}{1-\beta_1} < \left[(3{\cdot}5+1)^3\theta^{0{\cdot}5}\left(-\xi^2\frac{d\theta}{d\xi}\right)^2\right]_{\max} \times 0{\cdot}0707.$$

From Comrie's Tables the value of $[\ldots]_{\max}$ is about 143. Hence $\beta_1 < 0{\cdot}925$. Hence from the observed radius r_1, in conjunction with (10), V_1 must be of the order of 10^3. Then, again from Comrie's Tables, ξ_1 would have to be 9·5. For $\xi_1 = 9{\cdot}0$, $\theta_1^{0{\cdot}5}(-\xi^2 d\theta/d\xi)_1^2$ is about 0·389, or $(3{\cdot}5+1)^3\theta_1^{0{\cdot}5}(-\xi^2 d\theta/d\xi)_1^2$ is about 35·45. This gives from (15) $\beta_1 = 0{\cdot}82$. For $\xi_1 = 9{\cdot}0$, V_1 is 80·09, and r_1 comes out too big at $1{\cdot}45 \times 10^{13}$ cms. For $\xi_1 = 9{\cdot}3$, $\theta_1^{0{\cdot}5}(-\xi^2 d\theta/d\xi)_1^2$ is about 0·2535, $(3{\cdot}5+1)^3$ times this is 23·1, and β_1 comes out as 0·777. For $\xi_1 = 9{\cdot}3$, V_1 is 181·98, and r_1 comes out at $5{\cdot}5 \times 10^{12}$ cms.* For calculating β_1, this is perhaps near enough to the observed value $0{\cdot}9 \times 10^{12}$ cms. The value of p_1 is then

$$\frac{0{\cdot}777}{0{\cdot}223} \times 2{\cdot}547 \times 10^{-15} \times (5300)^4 = 7{\cdot}0 \text{ dynes cm.}^{-2}.$$

I should have preferred to adopt Jeans's value $n = 3{\cdot}25$,† but tables do not exist for this value of n; the pressure would have come out bigger. The main point is that we have a practicable method of estimating p_1 without recourse to photospheric opacity. The linkage of this with photospheric-opacity determinations of p_1 will be the subject of the next paper.

16. *Summary.*—A pair of equations, (9) and (10), are derived from which the photospheric pressure in any polytropic star can be calculated without knowledge of the photospheric absorption coefficient. They express the physical fact that the photospheric pressure holds down the mass M to its observed photospheric radius r_1, and they provide a reasonably accurate method of treatment of stellar boundary conditions. No assumption concerning the interior of the star is made save a general formula for its compressibility. The values of the photospheric pressure so calculated agree with those calculated from opacity estimates in the case of the Sun, and are of the right order of magnitude for all stars. A method is devised for ascertaining the mean polytropic index of any star from observed data concerning M, r_1, T_1 and p_1, and it is shown that the Sun must be built very nearly according to $n = 3$, in accordance with Eddington's views.

* For $n = 3{\cdot}5$, $\xi_0 = 9{\cdot}536$. This shows the "lack of completeness" of the polytrope required.

† *Astronomy and Cosmogony* (1928), p. 88.

POLYTROPIC EQUILIBRIUM. III. THE LUMINOSITY OF A STAR.

E. A. Milne.

1. It is the object of the present paper to combine the formula for the photospheric pressure in a star derived in the preceding paper from interior considerations with the standard formulæ for photospheric pressure based on calculations of the opacity in a stellar atmosphere. The result is to derive formulæ for the luminosities of stars in terms of their masses and effective temperatures of a radically different kind from those hitherto accepted.

2. We begin by restating the formulæ (9) and (10) of the preceding paper in the same notation :

$$p_1 = \frac{(R/\mu)^4 T_1^4}{4\pi G^3 M^2 \beta_1^3}(n+1)^3 \theta_1^{n-3}\left(-\frac{\xi^2 d\theta}{d\xi}\right)_1^2, \tag{1}$$

$$r_1 = \frac{\beta_1 GM}{(R/\mu)T_1}\frac{1}{n+1}\frac{\theta_1}{\left(-\xi\frac{d\theta}{d\xi}\right)_1}. \tag{2}$$

We proceed to calculate p_1 on various hypotheses as to the opacity of a stellar atmosphere and to equate the result to the p_1 given by (1), where, it should be remembered, ξ_1 is a parameter which is the solution of (2).

3. For the equilibrium of the layers above the photosphere we have the hydrostatic equation

$$\frac{dP}{dh} = g\rho,$$

and for the optical depth τ down to level h (measured inwards) we have

$$\frac{d\tau}{dh} = \kappa\rho,$$

where κ is the mass-absorption coefficient at level h. Hence the well-known equation

$$\frac{d\tau}{dP} = \frac{\kappa}{g}.$$

At the outer boundary of the atmosphere, where $p = p_0 = 0$, we have $P = P_0 = \frac{1}{3}aT_0^4 = \frac{1}{6}aT_1^4$. Hence integrating through the layers external to the photosphere down to the photospheric level $\tau = \frac{2}{3}$ we have

$$\frac{2}{3} = \frac{1}{g}\int_{P_0}^{P_1}\kappa dP = \frac{P_1 - P_0}{g}\bar{\kappa}_1, \tag{3}$$

where $\bar{\kappa}_1$ is the mean value of κ with respect to P through the layers external

to the photosphere. But $P_1 = p_1 + \frac{1}{3}aT_1^4$. Hence

$$\frac{2}{3}\frac{g}{\bar{\kappa}_1} = p_1 + \tfrac{1}{6}aT_1^4. \tag{4}$$

But

$$g = GM/r_1^2, \tag{5}$$

and the luminosity L is given by *

$$L = \pi a c r_1^2 T_e^4 = \pi a c r_1^2 T_1^4. \tag{6}$$

Hence, by (4),

$$\frac{p_1}{\frac{1}{3}aT_1^4} = \frac{2}{3}\frac{g}{\bar{\kappa}_1(\frac{1}{3}a)T_1^4} - \frac{1}{2} = \frac{2\pi cGM}{\bar{\kappa}_1 L} - \frac{1}{2}. \tag{7}$$

The expressions (7) are also equal to $\beta_1/(1-\beta_1)$. Hence solving for β_1,

$$\beta_1 = \frac{1 - \dfrac{\bar{\kappa}_1 L}{4\pi cGM}}{1 + \dfrac{\bar{\kappa}_1 L}{4\pi cGM}}, \tag{8}$$

or

$$L = \frac{4\pi cGM}{\bar{\kappa}_1}\frac{1-\beta_1}{1+\beta_1}. \tag{9}$$

4. The value of p_1 given by (7) is the photospheric pressure determined from the equilibrium of the outer layers in terms of their opacity, assuming radiative equilibrium for these layers. It must agree with the value given by (1). Inserting for p_1/T_1^4 from (7) in (1) and inserting for β_1 from (8) in (1) and (2) we find the explicit formulæ

$$\frac{4\pi cGM}{\bar{\kappa}_1 L}\frac{\left(1 - \dfrac{\bar{\kappa}_1 L}{4\pi cGM}\right)^4}{\left(1 + \dfrac{\bar{\kappa}_1 L}{4\pi cGM}\right)^3} = \frac{(R/\mu)^4}{2\pi G^3 M^2(\frac{1}{3}a)}(n+1)^3\theta_1^{n-3}\left(-\xi^2\frac{d\theta}{d\xi}\right)_1^2, \tag{10}$$

$$\left(\frac{L}{\pi a c T_1^4}\right)^{\frac{1}{2}} = \frac{GM}{(R/\mu)T_1}\frac{1 - \dfrac{\bar{\kappa}_1 L}{4\pi cGM}}{1 + \dfrac{\bar{\kappa}_1 L}{4\pi cGM}}\frac{1}{n+1}\frac{\theta_1}{\left(-\xi\dfrac{d\theta}{d\xi}\right)_1}, \tag{11}$$

where in the second equation we have substituted for r_1 from (6).

We may also write these equations in terms of β_1 considered now simply as a parameter,

$$\frac{\beta_1^4}{1-\beta_1} = \frac{(R/\mu)^4}{4\pi G^3 M^2(\frac{1}{3}a)}(n+1)^3\theta_1^{n-3}\left(-\xi^2\frac{d\theta}{d\xi}\right)_1^2, \tag{10'}$$

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* We here use (6) of the preceding paper. The equation is a consequence of $L = 4\pi r_1^2\sigma T_e^4$, $T^4 = \frac{1}{2}T_e^4(1+\frac{3}{2}\tau)$, $\sigma = \frac{1}{4}ac$, on putting $\tau = \tau_1 = \frac{2}{3}$, $T = T_1 = T_e$.

$$\left(\frac{L}{\pi acT_1^4}\right)^{\frac{1}{2}} = \frac{\beta_1 GM}{(R/\mu)T_1}\frac{1}{n+1}\frac{\theta_1}{\left(-\xi\frac{d\theta}{d\xi}\right)_1}, \qquad (11')$$

which must be united with (9).

5. If we eliminate ξ_1 between (10) and (11), or eliminate β_1 and ξ_1 between (10′), (11′) and (9), we obtain a relation between L, M and T_1 involving $\bar{\kappa}_1$, the mean opacity above the photospheric level $\tau = \frac{2}{3}$. This relation determines in principle the effective temperature $T_e(=T_1)$ to which the star adjusts itself so as to be able to radiate at the rate L, assuming only polytropic compressibility. In the steady state, the interior must be generating energy also at the rate L, which is thus the total source-strength. The value of $\bar{\kappa}_1$ will in general itself depend upon g and T_1, besides the intrinsic opacity of the material forming the photospheric layers. Thus to make use of the formulæ, unless $\bar{\kappa}_1$ is a given constant, we must express $\bar{\kappa}_1$ in terms of L, M and T_1 *via* (5) and (6).

We shall follow tradition, however, by regarding the (L, M, T_1) relation as enabling us to calculate L in terms of M and T_1. We must not, of course, commit the absurdity of supposing that L *depends upon* the photospheric opacity simply because this occurs in the relation ; it is the effective temperature which is controlled by this photospheric opacity, though this indeed, by reacting on interior conditions, may affect to some extent the rate of generation. Our system of equations (10) and (11) contains plenty of flexibility to permit adjustment to any law of internal generation, whatever the photospheric opacity ; for the index n is at our disposal. Provided the internal laws of energy-generation, opacity, equation of state, etc., are compatible with an over-all description by means of a polytropic index n, our equations hold good. The utility of our method is that, once n has been decided on, all these factors are dismissed from our consideration. I shall explain the general philosophical relation of our method to methods depending on detailed consideration of the interior in a later section. Some examples of explicit formulæ for $\bar{\kappa}_1$ will also be considered later. For the moment we consider (10) and (11) as they stand.

6. For all stars of not too high luminosity, β_1 is approximately unity (see the preceding paper). It is worth while, therefore, investigating the case $\beta_1 \sim 1$. Then (10) and (11) reduce to

$$L = \frac{8\pi^2 cG^4(\frac{1}{3}a)M^3}{\bar{\kappa}_1(R/\mu)^4}(n+1)^{-3}\theta_1^{-(n-3)}\left(-\xi^2\frac{d\theta}{d\xi}\right)_1^{-2}, \qquad (12)$$

$$L^{\frac{1}{2}} = \frac{(\pi ac)^{\frac{1}{2}}GMT_1}{R/\mu}\frac{1}{n+1}\frac{\theta_1}{\left(-\xi\frac{d\theta}{d\xi}\right)_1}. \qquad (13)$$

Eliminating the "sensitive" factor θ_1 between these,

$$\frac{\bar{\kappa}_1 L(R/\mu)^4}{8\pi^2 cG^4(\frac{1}{3}a)M^3}(n+1)^3\left(-\xi^2\frac{d\theta}{d\xi}\right)_1^2 = \left[\frac{(\pi ac)^{\frac{1}{2}}GMT_1}{(R/\mu)L^{\frac{1}{2}}}(n+1)^{-1}\left(-\xi\frac{d\theta}{d\xi}\right)_1^{-1}\right]^{n-3}. \qquad (14)$$

For the case $n=3$ we have simply

$$L = M^3 \frac{8\pi^2 cG^4(\frac{1}{3}a)}{\bar{\kappa}_1(R/\mu)^4}(n+1)^{-3}\left(-\xi^2\frac{d\theta}{d\xi}\right)_1^{-2}. \tag{15}$$

The adjustment to given values of L, M and $\bar{\kappa}_1$ is here carried out through the parameter ξ_1, but for nearly complete polytropes $(-\xi^2 d\theta/d\xi)_1^{-2}$ will have practically the value $(2{\cdot}018\ldots)^{-2}$. Hence

$$L \propto M^3/\bar{\kappa}_1. \tag{15'}$$

It is well known that the proportionality $L \propto M^3$ holds good * for the majority of the stars, this being an empirical formula describing the observed correlation of luminosity with mass. For dwarf stars of the main sequence, $\bar{\kappa}_1$ will be expected to be of the same order of magnitude for all stars, for the crude formula $\bar{\kappa}_1/g = \tau_1/p_1$, combined with the known similarity of the p_1 values amongst these stars and the slow variation † of g, gives a very slow change of $\bar{\kappa}_1$. We have thus not only practically established the observational proportionality $L \propto M^3$ on theoretical grounds if n may be taken as 3, but we have also obtained the constant of proportionality.

Let us test this formula (15). We best test it by employing it to obtain $\bar{\kappa}_1$ for the Sun, and then comparing the result with Chandrasekhar's values of κ for the Sun's photospheric layers calculated from the atomic opacity for hydrogen. To illustrate the enormous chances (15) has of going wrong I propose to write out the arithmetic in detail. Adopting $L\odot = 3{\cdot}78 \times 10^{33}$ ergs sec.$^{-1}$, $M\odot = 1{\cdot}985 \times 10^{33}$ grams, $\mu = \mu_H = 1$, $G = 6{\cdot}66 \times 10^{-8}$, $\frac{1}{3}a = 2{\cdot}547 \times 10^{-15}$, $R = 0{\cdot}826 \times 10^8$, we have

$$\bar{\kappa}_1 = \frac{8 \times (3{\cdot}1416)^2 \times (3 \times 10^{10}) \times (6{\cdot}66 \times 10^{-8})^4 \times 2{\cdot}547 \times 10^{-15} \times (1{\cdot}985 \times 10^{33})^3}{3{\cdot}78 \times 10^{33} \times (0{\cdot}826 \times 10^8)^4 \times 64 \times (2{\cdot}018)^2}$$

$$= \frac{9{\cdot}30 \times 10^{68}}{4{\cdot}58 \times 10^{67}} = 20{\cdot}3.$$

Chandrasekhar's tabulated value of $\kappa = \kappa_1$ at $\tau = \frac{2}{3}$ is about 166·8, and it varies through the exterior layers from 0 at $\tau = 0$ up to this value. The crude arithmetic mean $\bar{\kappa}_1$ is therefore of the order of 80, so that our value $\bar{\kappa}_1 = 20$ is discrepant by a factor of 4. This is roughly the same factor as that by which (11) (Paper II) overestimates p_1. If $n \neq 3$, (12) shows that the estimate of $\bar{\kappa}_1$ must be multiplied by the factor $\theta_1^{-(n-3)}$, which exceeds unity when $n > 3$. A slight increase in the value of μ above $\mu = 1$ would also improve the agreement.

Photospheric opacity is a clue to internal opacity only if the *same physical law* holds for the photosphere and interior. In particular, we are by no means entitled to conclude that the *interiors* of stars largely consist of hydrogen from the evidence of their masses, luminosities and effective

* Jeans, *Astronomy and Cosmogony*, 1928, p. 126, formula (118.1).

† See H. N. Russell, *Astrophys. Journ.*, **78**, 270, 1933, Table 7, where from type A2 to type *d*Mo, $\log_{10}(g/g\odot)$ only changes from $-0{\cdot}04$ to $+0{\cdot}19$.

temperatures, as has been done by certain writers from internal opacity calculations.

7. If $n \neq 3$, (14) gives the approximate proportionality, valid for $\beta_1 \sim 1$,

$$\bar{\kappa}_1 L^{1+\frac{1}{2}(n-3)} \propto M^n T_1^{n-3}. \tag{16}$$

E.g. for $n = 3{\cdot}5$,

$$\bar{\kappa}_1^{\frac{4}{5}} L \propto M^{\frac{14}{5}} T_1^{\frac{8}{5}}, \tag{16'}$$

and for $n = 4$,

$$\bar{\kappa}_1^{\frac{2}{3}} L \propto M^{\frac{8}{3}} T_1^{\frac{2}{3}}. \tag{16''}$$

We notice the fidelity with which L clings to $L \propto M^3$ even for these polytropes, assuming $\bar{\kappa}_1$ is reasonably constant from star to star.*

8. In the case of the "standard model" the opacity is the same everywhere, and $\bar{\kappa}_1$ coincides in value with the absorption coefficient for the interior. Our general formulæ (9), (10) and (11) provide a solution for this special case. If we are given L, M and $\bar{\kappa}_1$ *a priori*, equations (10) and (11) determine T_1 and ξ_1 for given n. The formulæ are readily applicable to the case of composite configurations, in which n is discontinuous in the interior. In this case θ is a non-Emden solution of Emden's differential equation, and the choice of the appropriate non-Emden solution will be made by supplementing (10) and (11) by a series of "equations of fit," the last zone (the central region) being described by an Emden solution. The difference from the treatment I have previously given for composite polytropes is that in (9), (10) and (11) the actual boundary conditions of a real star have been incorporated, instead of the unacceptable ones $T = 0$, $P = 0$.

9. In view of misunderstandings which have arisen in the past, it will be as well to consider what happens if the intrinsic opacity of the photospheric layers is supposed to change. For example, what would happen if the volatilisation of meteoric accretions altered the opacity? If the interior source-strength remained unchanged, the energy being delivered at the photospheric layers would be unchanged, but owing to their changed opacity a new linear temperature-gradient would tend to be set up; the first effect of the changed opacity at the old temperature gradient would be to tend to dam back the energy. There would therefore be required some adjustment of temperature in the photospheric layers, besides some adjustment due to the altered radiation pressure there. These would cause associated changes of radius and effective temperature, the total flux $4\pi r_1^2 \sigma T_e^4$ being ultimately the same as before. The photospheric pressure would be changed—the actual particles forming the level $\tau = \frac{2}{3}$ would be different—and the mass M enclosed inside the photosphere would be slightly, though negligibly, different. The net results for M and L constant could be readily calculated from (10) and (11), using some physical formula for $\bar{\kappa}_1$. The new configuration would have such an effective temperature that its photospheric layers would be capable, at the changed opacity, of radiating the same

* It is hardly necessary to remark that in the oft-quoted formula of the *current* theory, $L \propto M^{\frac{7}{5}}(1-\beta)^{\frac{3}{2}} T_e^{\frac{4}{5}}$, the factor $T_e^{\frac{4}{5}}$ has no observational support—the expression $L/M^{\frac{7}{5}}(1-\beta)^{\frac{3}{2}}$ does not, for actual stars, behave as $T_e^{\frac{4}{5}}$.

energy L as before. There would be reactions on the interior, and L, if the source-strength depended on physical conditions, would then also change. The new photospheric pressure would in the end be just the pressure necessary to hold the mass down to its new radius, though n might be changed also.

10. We now examine the effect of introducing a physical formula for $\bar{\kappa}_1$. The proper procedure would be to embark on a full-dress calculation on the same lines as those of my Bakerian lecture and Chandrasekhar's paper.* It may be worth while later to undertake this, but for present purposes a cruder calculation may suffice. Chandrasekhar adopts for κ the formula

$$\kappa = \frac{\alpha p_e \bar{x}}{T^{\frac{11}{2}}}, \tag{17}$$

where α is a physical constant and p_e is the electron pressure, related to p the gas pressure by

$$\frac{p_e}{p} = \frac{\bar{x}}{1+\bar{x}};$$

$\bar{x}$ is the average number of free electrons per atom. Thus

$$\kappa = \frac{\alpha p \bar{x}^2}{(1+\bar{x})T^{\frac{11}{2}}}. \tag{18}$$

We can now either calculate p_1 directly from this formula and insert it in (1), or follow the method of § 2, calculate $\bar{\kappa}_1$ and insert it in (7) as a function of g and T_1, thence in terms of M, L and T_1. I shall, however, avail myself of the elegant approximate treatment given in § 5 of Chandrasekhar's paper, which is certainly valid for all except possibly very luminous stars. Chandrasekhar uses a variable u, such that

$$p = u \frac{p'^{\frac{19}{16}}}{p_1'^{\frac{3}{16}}}, \tag{19}$$

where $p' = \frac{1}{3}aT^4$, $p_1' = \frac{1}{3}aT_1^4$. Let u_1 denote the value † of u at $\tau = \frac{2}{3}$. Then

$$p_1 = u_1 p_1'. \tag{20}$$

But, by his (23),

$$p' = p_0' \left(1 - \frac{u^2}{B^2}\right)^{-\frac{8}{16}}, \tag{21}$$

where

$$B^2 = \frac{16}{19}A = \frac{16}{19}\left[\frac{4\pi cGM}{L}\,\frac{1+\bar{x}}{\bar{x}^2 a(\frac{1}{3}a)}T_1^{\frac{3}{2}}\right]. \tag{22}$$

But by Schwarzschild's theorem, $p_1'/p_0' = 2$. Hence by (21)

$$\begin{aligned} u_1 &= B(1-(\tfrac{1}{2})^{\frac{19}{8}})^{\frac{1}{2}} \\ &= fB, \end{aligned} \tag{23}$$

say, where $f = 0{\cdot}898$.

* *M.N.*, **92**, 186, 1932.

† Not to be confused with Chandrasekhar's u_1, which is one of the roots of a certain quadratic.

Hence by (20) and (23)

$$\frac{\beta_1}{1-\beta_1}=\frac{p_1}{\frac{1}{3}aT_1{}^4}=f\left(\frac{16}{19}\right)^{\frac{1}{2}}\left[\frac{4\pi cGM}{L}\frac{1+\bar{x}}{\bar{x}^2\alpha(\frac{1}{3}a)}T_1{}^{\frac{3}{2}}\right]^{\frac{1}{2}}. \tag{24}$$

Inserting this in (1) we get

$$f\left(\frac{16}{19}\right)^{\frac{1}{2}}\left[\frac{4\pi cGM}{L}\frac{1+\bar{x}}{\pi^2\alpha(\frac{1}{3}a)}T_1{}^{\frac{3}{2}}\right]^{\frac{1}{2}}=\frac{(R/\mu)^4}{4\pi G^3M^2\beta_1{}^3}(n+1)^3\theta_1{}^{n-3}\left(-\xi^2\frac{d\theta}{d\xi}\right)_1^2. \tag{25}$$

Eliminating the "sensitive" factor θ_1 between (25) and (11′) we get

$$f\left(\frac{16}{19}\right)^{\frac{1}{2}}\left[\frac{4\pi cGM}{L}\frac{1+\bar{x}}{\bar{x}^2\alpha(\frac{1}{3}a)}T_1{}^{\frac{3}{2}}\right]^{\frac{1}{2}}\frac{4\pi G^3M^2\beta_1{}^3}{(R/\mu)^4}(n+1)^{-3}\left(-\xi^2\frac{d\theta}{d\xi}\right)_1^{-2}$$
$$=\left[\frac{L^{\frac{1}{2}}(R/\mu)}{(\pi ac)^{\frac{1}{2}}T_1\beta_1GM}(n+1)\left(-\xi\frac{d\theta}{d\xi}\right)_1\right]^{n-3}. \tag{26}$$

In this formula β_1 is to be given its value according to the extreme members of equality (24).

We shall approximate by taking $\beta_1\sim 1$. Then (26), for ξ_1 in the neighbourhood of ξ_0, gives the approximate proportionality

$$L^{\frac{1}{2}+\frac{1}{2}(n-3)}\propto M^{\frac{5}{2}+(n-3)}\,T_1{}^{\frac{3}{4}+(n-3)}$$

or

$$L^{n-2}\propto M^{2n-1}T_1{}^{2n-\frac{9}{2}}. \tag{27}$$

E.g.

for $n=3$, $\quad L\propto M^5T_1{}^{\frac{3}{2}}$; (27$^{\text{i}}$)

for $n=3{\cdot}5$, $\quad L\propto M^4T_1{}^{\frac{5}{3}}$; (27$^{\text{ii}}$)

for $n=4$, $\quad L\propto M^{\frac{7}{2}}T_1{}^{\frac{7}{4}}$; (27$^{\text{iii}}$)

for $n=5$, $\quad L\propto M^3T_1{}^{\frac{11}{6}}$. (27$^{\text{iv}}$)

The results for $n=3$ and $n=3{\cdot}5$ seem to give too rapid a change with M. As for the dependence on T_1, it must be remembered that $\bar{x}$ in (26) depends on T_1 and $(p_e)_1$, and $(p_e)_1$ will itself depend on g and so on L, M and T_1. On the whole, (26) and the set (27) give less insight into stellar luminosity than the corresponding formula (14) and set (16), which contain the unspecified $\bar{\kappa}_1$. It is probable that the last word has not been said on the calculation of photospheric opacity, even with Chandrasekhar's refinements. Formulæ (10) and (11), or the approximate formula (14), are in a form ready for immediate use with any future improvement in photospheric opacity calculations.

11. It is now necessary to explain how it comes about that we have been able to make a calculation of stellar luminosity without any assumptions about the interior save an assumption as to its general compressibility, *i.e.* an assumption (expressed by choice of n) as to its general density-distribution. I explained the general principles in my 1929 paper on "The Masses, Luminosities and Effective Temperatures of the Stars," * but the

* *M.N.*, **90**, 17, 1929.

introduction to that paper has been generally disregarded or misunderstood, and I will try to expound the natural philosophy of the subject in clearer language.

If we knew the physical nature of the mechanism of the generation of stellar energy, we could straightforwardly calculate all the characteristics of a star, including its luminosity, in terms of the single parameter M and the ultimate chemical constitution, as Vogt and Russell have emphasised.

In the absence of such a knowledge there are two distinct methods of attempting to calculate, not the luminosity in terms of the mass, but the relation between mass, luminosity and effective temperature. We can replace our ignorance of interior conditions either by an assumption about the geographical distribution of sources of energy (method 1), or by an assumption about the geographical distribution of density (method 2). In method 1 the sources total to some given (observed) value L, and the problem is to calculate the external radius of the given mass M endowed with sources of energy totalling to L distributed in the assumed way. Assumptions about the opacity law and the equation of state are required. In the course of carrying out the radius calculation we, of course, determine the associated density-distribution. In method 2 we arrange the given mass M, of given (observed) radius r_1, in mechanical equilibrium according to the assumed density-distribution, and calculate the rate at which this mass, so arranged, radiates to space. If the mass were left unprovided with sources of energy, it would of course be changing in temperature at each point. We therefore calculate the strength and distribution of energy-sources necessary to maintain the temperature at each point constant in time. These necessarily total to the star's radiation to space at its surface, already calculated, so that if we are only interested in the luminosity there is no need to determine the heat losses (*i.e.* the corresponding source-strengths) in the interior.

In method 1 our ignorance is conveniently replaced by some hypothesis about ϵ, the relative source-distribution, or, better, Eddington's variable η, the ratio of $L(r)/M(r)$ to L/M. In method 2 our uncertainty is conveniently replaced by some hypothesis about n, the mean polytropic index, which is a measure of the compressibility and determines the concentration of matter towards the centre. In method 1 we may calculate n from the assumptions concerning η. In method 2 some discussion of probable interior conditions furnishes a value for n, and we can later calculate η. Methods intermediate between the two can of course be followed, but the above describes the broad difference of principle between the methods that can be followed. The two methods must of course agree for any real star, and this allows one to test the compatibility of the alternative forms of hypothesis used to replace our ignorance.*

* When I pointed out these considerations before, Eddington suggested that it was as though there were "rival theories as to what is the correct solution of a quadratic equation" (*M.N.*, **90**, 280, 1930). There are not two theories, but two methods—just as there are two methods for solving a quadratic equation, one by factorisation and one by completing the square, and that the two methods are consistent is one of the most important theorems of algebra.

In method 2 we actually calculate the actual radiation of the surface layers, as arranged, to space, and we accordingly need only a detailed knowledge of the surface layers. This is the method of the present paper. In method 1 we must take care that the surface layers are in a condition to radiate to space the given amount of energy being generated in the interior. It is rightly stated that the surface layers can always adjust themselves to this effect. But as the problem in this case is precisely to calculate the radius for given L and M, the problem is not finished until we have calculated how this adjustment takes place. For the matter near the boundary is at a low density, that is, a given amount of matter here is occupying a large volume, and the radius, in other words the total volume, of the star will be very sensitive to the precise way in which the adjustment takes place. How sensitive the radius in fact is, follows from a glance at (2) above, from the small, "sensitive" factor θ_1. Anyone who has analysed the "standard model" with the correct boundary conditions for radiative equilibrium is aware of the extreme sensitiveness of the radius in this case—a consequence of the neutral character of the equilibrium of the complete polytrope.

12. Even on the conventional analysis of the standard model, with boundary conditions $T=0$, $P=0$, the L, M, κ relation is of the character found in this paper. To make any use of this model, we must calculate the radius, *i.e.* extend the solution to the boundary. Since the assumption underlying this model is $\kappa\eta=$constant, for consistency's sake we must carry through this assumption in all contexts if in one. To calculate the radius from the properties of the Emden polytrope (or to calculate central values from the radius, which is what is required) we must extend the assumption $\kappa\eta=$const. to the boundary. Then, since $L(r)=4\pi cGM(r)(1-\beta)/\kappa\eta$, since at the boundary $\eta=1$, $L(r)=L$ and $M(r)=M$ we get $L=4\pi cGM(1-\beta)/\kappa_1$, where κ_1 is the boundary opacity; κ_1, in other words, adjusts itself to the given L and M.* If in this crude calculation we replace the unacceptable boundary conditions $T=0$, $P=0$ by the actual boundary conditions of a real star, we arrive at formulæ (9), (10), (11) of this paper. It is no use objecting that $\kappa\eta$ is not supposed to remain constant up to the boundary, for it is *in fact* assumed constant, in constructing the model, right up to the boundary—in expressing T_c, etc., in terms of r_1. If it does not in fact remain constant up to the boundary, then the radius calculation breaks down, and the actual radius may be widely different; and so the central temperature, etc., will not be given by inserting the value of the *actual* radius in the formulæ obtained by supposing $\kappa\eta=$const. up to the boundary.

13. I now briefly survey the course of my researches in stellar structure during the last few years.† In *Handbuch der Astrophysik*, Bd. III/1, 1930, I gave a faithful account of Eddington's researches, in accordance with my

* If κ_1 does not have this value, the layers near the boundary will be heating up or cooling down.

† My reason for so doing is that I have been accused of "readjusting my views from time to time as new evidence is obtained" (Eddington, *Zeits. für Astrophys.*, 3, 140, 1931). Naturally my views develop with the research, but Eddington seemed to imply a rather discreditable inconsistency.

editor's instructions. But in the course of doing so I lost confidence in some of Eddington's results, and so informed my editor. I contented myself with adding a section of my own on "Homologous Stars in Radiative Equilibrium" (based largely on the work of Vogt *), at the end of which I wrote (p. 234): "It would appear from this analysis that the luminosity of the star fixes, not the mean value of κ through the interior, but the limit to which κ approaches as we approach the boundary from the interior." This was written before my 1929 paper in *Monthly Notices*. In the latter paper I gave a derivation of formula (10′) above in a slightly modified form, without assuming anything about the interior save the general form of the density-distribution, but assuming radiative equilibrium and the gas-laws in the photosphere. I did not, however, derive the complementary equation (11′), without which (10′) is essentially incomplete. However, in the introduction to this paper I wrote: ". . . I investigate the relations between the masses, luminosities and effective temperatures of the stars from a standpoint which is philosophically different from that adopted by Professor Eddington in his well-known writings. The main conclusions are that it is not possible to infer from the observed masses, luminosities and temperatures that the interiors of stars are necessarily composed of perfect gas; and that it is not possible to deduce the value of the absorption coefficient for the stellar interior. Instead we are led to infer a single definite fact concerning the internal density-distribution." By "a standpoint philosophically different" from Eddington's I meant method 2, as here expounded (Eddington † uses method 1). The "single definite fact" concerning the density-distribution has been given precise form in the present series of papers by the evidence that n is close to 3 (compressibility agreeing with Eddington's theory) and by the outline of a method for determining n for any observed star.

With regard to the occurrence of photospheric opacity in the relation connecting L, M and T_e, this was such a breach with tradition that I realised that it would be impossible to convince astronomers of this unless I could construct actual examples containing different opacities in interior and exterior layers, and could show that the control was exercised by the opacity in the latter. Unfortunately no analytical treatment involving a *formula* for κ in terms of ρ and T could possibly establish this if we use the one and the same formula right through the star. If, for example, we assume κ constant, and then determine κ "astronomically" from an observed L, M and T_e, we cannot say whether the resulting value applies to the exterior layers or the interior, for the assumption is already that κ is the same in both. Again, if we adopt $\kappa = \alpha\rho/T^{\frac{7}{2}}$, and then determine α "astronomically," we again cannot say whether the resulting α applies to photosphere or interior, for

* *A.N.*, **226**, 301, 1926; **230**, 241, 1927. See in particular his paper (*A.N.*, **233**, 13, 1928, footnote 1) where he explicitly shows that the fundamental relations connecting the luminosity with the other characteristics of the star depend on the equation of state applied to the outer layers only.

† In principle, though in some ways his method is intermediate, as he takes given M and r_1 (method 2) and given η (method 1).

we have already assumed the same α to hold through the star. Formulæ do not "smell." I therefore tried models with a discontinuous κ, one value κ_1 holding in the outer layers and another κ_2 holding in the interior. But I did not succeed in putting my solutions into a form that could be applied to illustrate the point in question.

Meanwhile there was the other problem—of deducing the interior density-distribution from surface data. Formula (10′) with $n=3$ was sufficiently enticing in itself, and I accordingly applied it to Eddington's "standard model" under the definite simplifying conditions $\kappa=$const., $\epsilon=$const. In doing so I was imposing the condition of consistency of methods 2 and 1, formula (10′) representing the result of an application of method 2, and "standard model" conditions being a case of method 1. The result was immediately to give the classification of types of density-distribution into "collapsed," "centrally condensed" and "Emden" solutions according to the value of a discriminant C, which was none other than the dimensionless factor on the right-hand side of (10′). This led, following R. H. Fowler's researches,* to the study of configurations with degenerate cores, to a satisfactory theory of white dwarfs, to suggestions about novæ and Cepheids, and prompted various writers † to work out more fully the theory of the degenerate gas. The ultimate object in method 1 is always to calculate the radius r_1 in terms of L, M, opacity and molecular weight, and this series of researches, as far as I had contemplated it, was completed by Fairclough in a paper this year.‡ In the meantime the study of composite configurations had been carried much further by Chandrasekhar.§

The general result of the radius calculations was that the radius came out always very small, of not much more than white-dwarf order. Hence either stars were not composite configurations with degenerate cores, or else something was wrong with the calculations. The only remaining unsatisfactory part of the analysis of the standard model was the treatment of boundary conditions. In pursuing method 1, I had already tried a better treatment of boundary conditions in another paper,‖ and this had suggested the possibility of much larger radii for *gaseous* configurations than the radii of observed stars. Though this possibility had been unfavourably regarded by other writers,¶ their results were to me inconclusive as they did not fully embody in their solutions the actual stellar boundary conditions, and it was anyhow clear that the imposition of a boundary condition $T_0=0$ must enormously extend the gaseous outer layers. Was it not possible that a really sound treatment of boundary conditions would be illuminating? I therefore commenced, at first purely as a formal problem, the study of various kinds of configurations *under external pressure*, for example the

* *M.N.*, **87**, 114, 1926.
† Chandrasekhar, Kothari, Majumdar, Miss Swirles and others.
‡ *M.N.*, **95**, 585, 1935.
§ *M.N.*, **91**, 456, 1931 ; **95**, 207, 226, 676, 1935.
‖ *Zeits. für Astrophys.*, **4**, 75, 1932, where method 1 is expounded in detail.
¶ Cowling, *Zeits. für Astrophys.*, **4**, 331, 1932 ; ten Bruggencate, *ibid.*, **9**, 110, 1934.

standard model and isothermal configurations. It appeared possible to calculate the surface pressures required to "hold down" aggregates of stellar mass to the stellar order of radius, and these came out, in many cases, to my great interest, of the photospheric order. I then undertook a systematic study of polytropes under external pressure, the results of which are summarised in I of the present series of papers. In seeking to apply these results to real stars, I arrived at equations (1) and (2) of this paper. The object, of course, at that stage was to calculate r_1 in terms of p_1. But the order of magnitude of p_1 was independent of r_1, for n close to 3, and of the photospheric order. This was astonishing, for one was well accustomed, in studying stellar atmospheres, to regard the photospheric pressure as conditioned by the photospheric opacity. I slowly realised the double rôle of photospheric pressure as the pressure down to which we see and as the pressure holding down the compressible mass to its observed radius. I then noticed that by forcing in the radiation constant on the right-hand side of (1), it could be made to assume the form (10′), which in one form or another had been the starting-point and connecting link in all the researches mentioned. The starting-point had been regained. A model had been constructed satisfying the boundary conditions of nature in which only the surface opacity was relevant to the relation between L, M and T_e—the problem that had defeated me after my 1929 paper. Re-reading the 1929 paper, I found that in essentials (though not in all details) it correctly described the new position to which I had been led, which was in effect the original position.*

It is a pleasure to express my thanks to Professor H. H. Plaskett, who has kindly discussed with me the present results at all stages, and who suggested their mode of arrangement.

19 *Northmoor Road,*
Oxford:
1935 *December.*

* Since the above was written it has been found possible to obtain the main results of this paper without the assumption of polytropic compressibility, by completing the method followed in the 1929 paper by the derivation of the generalisation of (2) or (11′).

ON STELLAR CONFIGURATIONS WITH CORE POSSESSING A HIGH DENSITY OF ENERGY.

A. B. Severny.

(Communicated by E. A. Milne)

Introduction.—In the well-known investigations on the internal constitution of stars carried out by A. S. Eddington, we are dealing with the model of a star consisting entirely of matter in the condition of a perfect gas. There can, of course, be no question whether the matter in the outer layers of the star is in a state other than that described by the laws of a perfect gas. However, the analysis of this *eigenwert* problem of equations of equilibrium made by Professor E. A. Milne * allowed the conclusion that in the interior of stars there may, in general, be regions occupied by matter with so high a density that in these regions the laws of perfect gases do not hold. Milne gave a general theory of these two-phase configurations (perfect gas/degenerate gas) which was later developed in the works of S. Chandrasekhar,† Cillié ‡ and others. In these investigations the second phase is represented by matter in the condition of degeneracy, *i.e.* in the condition depicted in the statistics of Fermi-Dirac. E. Stoner § and Chandrasekhar || have occupied themselves also with the statistic equilibrium of a gas at high temperatures, *i.e.* with consideration of the relativistic effect.

The recent progress of the physics of the nucleus, however, is suggesting (compare P. M. S. Blackett ¶) that in the interior of stars there may exist regions of matter with great density of energy, where photons with energy $h\nu >> 2m_0c^2$ in a state of statistical equilibrium are being generated by interactions between nuclei pairs, consisting of a positron and an electron, possessing the highest possible energy, as follows from Dirac's theory of the positron.

As matter in this state possesses a series of properties interesting to the astrophysicist (annihilation of the positron and of the electron, reaction of the positron with the nucleus and the reverse process of photoelectric formation of pairs), this supposition seems quite rational from the point of view of the problem of the origin of the generation of energy in stars, whereas in the treatment of a star as a perfect gas sphere, it remains to a certain degree indeterminate.

Therefore the aim of the present paper is as follows : arising from the above supposition (*a*) to form an equation of the state of matter at a high density of energy in a form convenient for the study of the problem of equilibrium, and (*b*) arising from Milne's theory of two-phase configurations —to investigate approximately the general character of the model.

* *M.N.*, **91**, 4, 1930 ; **92**, 610, 1932.
‡ *Ibid.*, **93**, 574, 1933.
|| *Ibid.*, **91**, 446, 1931.
† *Ibid.*, **91**, 456, 1930.
§ *Ibid.*, **92**, 662, 1932.
¶ *Nature*, **132**, 919, 1933.

I. *Equation of State of Matter at a High Density of Energy* *

It may be mentioned that the statistical equilibrium of a similar gas has been discussed in the works of Mrs. G. Wattagin † and M. Bronstein.‡ The former considers the total charge of the mixture of positrons and of electrons to be different from zero, which cannot be the case in the interiors of stars. The second author has not reached in his researches the stage where the results could be applied to investigations of a macroscopic character.

Let us discuss a mixture consisting of n photons, n' electrons and n'' positrons in statistical equilibrium, provided the total charge of the mixture is equal to zero, *i.e.*

$$\sum_i (n_i' - n_i'') = 0. \tag{1.1}$$

The total energy of the mixture is

$$U = \sum_i (n_i \epsilon_i + n_i' \epsilon_i' + n_i'' \epsilon_i'') = \text{const.}, \tag{1.2}$$

where ϵ_i is the corresponding energy level. The probability of the distribution without interaction of the particles will be

$$\prod_i \frac{(n_i + g_i - 1)!}{n_i!\,(g_i - 1)!} \frac{g_i'!}{n_i'!\,(g_i' - n_i')!} \frac{g_i''!}{n_i''!\,(g_i'' - n_i'')!},$$

where g_i is the number of cells in the *i-th* level in the four-dimensional phase-space. The condition for the maximum of entropy is

$$\delta \lg W = \sum_i [\lg(n_i + g_i) - \lg n_i]\delta n_i + \sum_i [\lg(g_i' - n_i') - \lg n_i']\delta n_i + \sum_i [\lg(g_i'' - n_i'') - \lg n_i'']\delta n_i'' = 0,$$

whence for the photons we get Planck's formula

$$n_i = \frac{g_i}{e^{\beta \epsilon_i} - 1}, \qquad \beta = \frac{1}{kT}, \tag{1.3}$$

and for the electrons and positrons we get a Fermi-Dirac distribution with zero chemical potential given by

$$n_i' = \frac{g_i'}{e^{\alpha + \beta \epsilon_i'} + 1}, \qquad n_i'' = \frac{g_i''}{e^{-\alpha + \beta \epsilon_i''} + 1}, \qquad \alpha = 0. \tag{1.4}$$

If $kT >> m_0 c^2$, then the number of cells g_i' and g_i'' will be determined by

$$\frac{8\pi}{(ch)^3} (\epsilon_i^2 - m_0^2 c^4)^{1/2} \epsilon_i d\epsilon_i \tag{1.5}$$

as the impulse is $p = \left[\left(\frac{\epsilon}{c}\right)^2 - (m_0 c)^2\right]^{1/2}$ according to relativistic mechanics.

* In the formulation of this part the author is indebted to Prof. I. E. Tamm.

† *Phil. Mag.*, **17**, 910, 1934.

‡ *Comm. of t. Ac. of Sc. U.S.S.R.*, v. ii, N. 8, 1934.

However, for the calculation of the total number of positrons and of electrons per unit of volume we can consider $p \simeq \frac{\epsilon}{c}$, so that the number of cells in the *i-th* quantum level will be

$$\frac{8\pi}{h^3} p_i^2 dp_i = \frac{8\pi}{(ch)^3} \epsilon_i^2 d\epsilon_i. \tag{1.6}$$

Then, with the aid of (1.4) and (1.6) for the total number of particles per unit of volume we get the following expression :—

$$N = \frac{16\pi}{(ch)^3} (kT)^3 \int_0^\infty \frac{\xi^2 d\xi}{e^\xi + 1}, \tag{1.7}$$

and taking into account (1.5) for the energy density of the positron-electron mixture, strictly speaking, we find the expression

$$U_1 = \frac{16\pi}{(ch)^3} (kT)^4 \int_{\frac{m_0 c^2}{kT}}^\infty \frac{\xi^2 \left[\xi^2 - \left(\frac{m_0 c^2}{kT}\right)^2\right]^{1/4}}{e^\xi + 1} d\xi. \tag{1.8}$$

However, the integral in the right-hand side of this formula practically does not depend on T, and we shall accordingly consider it as constant. As the pressure P is $\frac{1}{3}U_1$, eliminating kT between (1.7) and (1.8) we obtain

$$P = \frac{1}{6} \frac{hc}{(2\pi)^{1/2}} N^{4/3} \phi, \tag{1.9}$$

where

$$\phi = \int_{\frac{m_0 c^2}{kT}}^\infty \frac{\xi^2 \left[\xi^2 - \left(\frac{m_0 c^2}{kT}\right)^2\right]^{1/2}}{e^\xi + 1} d\xi \left[\int_0^\infty \frac{\xi^2 d\xi}{e^\xi + 1}\right]^{4/3}.$$

It is of interest to note at the same time the following: If the density of this matter is such that the mean distance between the particles is of the order of their radii $e^2/m_0 c^2$, then the number of particles per unit of volume will be of the order 10^{39} and the mean energy of one particle $\frac{U_1}{N} \propto N^{1/3}$ will be $\sim 10^9 V$ which is of the order of magnitude of the energy of cosmic rays. Or else—at the annihilation of the positron and of the electron one or two photons can spring up with a wave-length $\sim 10^{-14}$ cm.

Thus, with the densities discussed, the behaviour of photons and of particles is *sui generis* "indistinguishable." A further increase of density will lead us probably directly into the domain of the relativistic quantum theory.

The ultimate problem consists in obtaining an equation of state of the form $P = P(\rho, T)$, where ρ is the density in the system of co-ordinates with regard to which matter rests as a whole (in Eddington's * designation ρ_{00}), and this density is different from the proper density ρ_0.

* *The Math. Theory of Relativity*, § 46.

As the density is determined by the gravitational interaction of particles, *i.e.* by $\Delta\Omega = G_{\mu\nu}$, where Ω is Newton's potential at the given point, and as this interaction depends on the energy of particles, it follows from the fundamental relation $-8\pi T_{\mu\nu} = G_{\mu\nu} - \frac{1}{2}g_{\mu\nu}G$ that

$$\rho = T_{44}, \tag{1.10}$$

i.e. this density is a principal component of the energy-tensor when the distribution of matter is steady. T_{44} for Galileian values $g_{\mu\nu}$ will represent the quantity of mass (or of energy) per unit of volume

$$T_{44} = \frac{E}{c^2}N = \frac{p}{c}N,$$

where E and p is the mean energy and the impulse of the particle respectively, and N is the number of particles per unit of volume equal to the double number of cells of the phase-space in the unit of volume. (In our case the relation of the number of cells to the number of particles will be the same, as regards order of magnitude, as for photons, *i.e.* $\sim e^{\alpha} = 1$.)

It follows that $p = (3/8\pi)^{1/3}hN^{1/3}$, whence

$$\rho = T_{44} = (3/8\pi)^{1/3}\frac{h}{c}N^{4/3}$$

or

$$N^{4/3} = (8\pi/3)^{1/3}\frac{c}{h}\rho. \tag{1.11}$$

Substituting $N^{4/3}$ from (1.11) into (1.9) we get the equation of state in the form

$$P = (1/6)^{1/3}\phi\tfrac{1}{3}c^2\rho \equiv K_3\rho, \tag{1.12}$$

where

$$K_3 = \tfrac{1}{3}(1/6)^{1/3}\phi c^2. \tag{1.13}$$

II. *Problem of Equilibrium*

When investigating this problem we will make the following assumptions: (1) The star represents a two-phase configuration—(*a*) the envelope is a perfect gas with an equation of state $p = \frac{R}{\mu}\rho T$, where p is the pressure, ρ the density, and T is the temperature of the gas, the molecular weight μ being considered constant; (*b*) the core is matter with the equation of state (1.12). (2) For the model under discussion there are different integrals of pressure in the two phases, *i.e.* the ratio of the pressure of gas p to the pressure of radiation p' is taken for each phase according to the formulæ

$$(a)\ p = \frac{\beta_1}{1-\beta_1}p', \qquad (b)\ p = \frac{\beta_2}{1-\beta_2}p' + D. \tag{2.1}$$

Here

$$\beta_1 = 1 - \frac{k_1 L}{4\pi cGM}, \qquad \beta_2 = 1 - \frac{k_2 L}{4\pi cGM}, \tag{2.2}$$

where k_1 and k_2 are the products of the mean values of the coefficient of opacity for the whole star and of the mean value η (which value determines the concentration of the energy-sources towards the centre), and D is a constant.

For the envelope the equations of the problem

$$\frac{d}{dr}(p+p') = -\frac{GM(r)}{r^2}\rho, \qquad \frac{d}{dr}p' = -\frac{k_1 L(r)}{4\pi c r^2}\rho \tag{2.3}$$

are reduced to Emden's equation with index $n=3$,

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^3, \tag{2.4}$$

giving

$$M(r) = -\frac{4}{(\frac{1}{3}a\pi G^3)^{1/2}}\left(\frac{R}{\mu}\right)^2 \frac{(1-\beta_1)^{1/2}}{\beta_1^2}\xi^2\frac{d\theta}{d\xi} \tag{2.5}$$

with the aid of the well-known substitutions

$$T = \lambda\theta, \qquad r = \frac{\xi}{\lambda_1}\frac{1}{(\frac{1}{3}a\pi G)^{1/2}}\frac{R}{\mu}\frac{(1-\beta_1)^{1/2}}{\beta_1}, \tag{2.6}$$

where ξ and θ are non-dimensional variables, and λ_1 is as yet arbitrary. The function $\theta(\xi)$ satisfies the initial conditions that at $r=r_1$:

$$\xi = \xi_1, \qquad \theta(\xi_1) = 0, \qquad -\left[\xi^2\frac{d\theta}{d\xi}\right]_{\xi=\xi_1} \equiv \omega_3 = C^{-1/2}, \tag{2.7}$$

where

$$C = \frac{16(R/\mu)^4}{\pi(\frac{1}{3}a)G^3M^2}\frac{1-\beta_1}{\beta_1^4} \tag{2.8}$$

is the discriminant. Of course the solution $\theta(\xi)$ holds good only in the interval $\xi' < \xi < \xi_1$, in virtue of the existence of another phase for $\xi < \xi'$.

Taking into account of (2.1) for the core with equation of state (1.12) the equation of mechanical equilibrium,

$$\frac{K_3}{G\beta_2}\frac{d}{dr}ln\rho = -M(r)/r^2, \tag{2.9}$$

is reduced to Emden's equation for isothermal gas-spheres,

$$\frac{1}{\eta^2}\frac{d}{d\eta}\left(\eta^2\frac{d\psi}{d\eta}\right) = -e^{\psi}, \tag{2.10}$$

giving

$$M(r) = -\frac{1}{(4\pi)^{1/2}}\frac{1}{\lambda_2^{1/2}}(K_3/G\beta_2)^{3/2}\eta^2\frac{d\psi}{d\eta} \tag{2.11}$$

with the aid of the substitutions :

$$\rho = \lambda_2 e^{\psi}, \quad r = (K_3/4\pi G\beta_2)^{1/2}\frac{1}{\lambda_2^{1/2}}\eta, \tag{2.12}$$

where η and ψ are non-dimensional, and λ_2 is as yet arbitrary. The solution of (2.10) may be selected so as to give at $r=0$:

$$\eta=0, \qquad \rho=\rho_c, \qquad \psi(\eta)=0, \tag{2.13}$$

and at $\eta=\eta'>0$ the quantities (2.11) and (2.12) ought to give the same values r' and $M(r')$ as (2.5) and (2.6) at $\xi=\xi'$, *i.e.* all non-dimensional variables ought to satisfy certain "equations of fit."

In order to prevent the arbitrary λ from entering these conditions of "fit" we will exclude them from the condition of continuity of the integral of pressure at the surface of demarcation, and will thus introduce into the expression of the radius of the configuration the values of the non-dimensional variables at the surface of demarcation. That is, approaching the surface of demarcation from the perfect-gas phase we get by (2.1),

$$\frac{R}{\mu}\rho' T' = \tfrac{1}{3}aT'^4\frac{\beta_1}{1-\beta_1},$$

and approaching from the "gas-phase" (1.12) we find similarly

$$K_3\rho' = \tfrac{1}{3}aT'^4\frac{\beta_2}{1-\beta_2}+D,$$

and therefore we obtain the interfacial density, temperature and D in the forms

$$\rho'=\frac{\frac{1}{3}a}{R/\mu}\left(\frac{K_3}{R/\mu}\right)^3\frac{\beta_1}{1-\beta_1}, \quad T'=\frac{K_3}{R/\mu}, \quad D=\tfrac{1}{3}a\left(\frac{K_3}{R/\mu}\right)^4\frac{\beta_1-\beta_2}{(1-\beta_1)(1-\beta_2)}. \tag{2.14}$$

It is of interest to note that the interfacial temperature is constant for all stars, built according to the examined model. From (2.6) and (2.12) we get

$$\left.\begin{aligned}\lambda_1&=\frac{K_3}{R/\mu}\cdot 1/\theta(\xi'),\\ \lambda_2&=\frac{\frac{1}{3}a}{R/\mu}\left(\frac{K_3}{R/\mu}\right)^3\frac{\beta_1}{1-\beta_1}e^{-\psi(\eta')}.\end{aligned}\right\} \tag{2.15}$$

Excluding λ_1 from (2.15) and (2.6) we find for the external radius of the configuration the following expression :—

$$r_1=r'\frac{\xi_1}{\xi'}=\frac{(R/\mu)^2}{K_3}\frac{1}{(\frac{1}{3}a\pi G)^{1/2}}\frac{(1-\beta_1)^{1/2}}{\beta_1}\theta(\xi')\xi_1. \tag{2.16}$$

To determine the external radius of the configuration (2.16) it is obviously necessary to know $\theta(\xi')$ and ξ', which satisfy the conditions of "fit." The latter are easy to get; if the surface of demarcation occurs at $r=r'$, then identifying with each other the values r' and $M(r')$ (2.6), (2.5) and (2.12), (2.11) all taken at $\xi=\xi'$, $\eta=\eta'$, we get the conditions of fit,

$$\left.\begin{aligned}\xi'\theta(\xi')&=\tfrac{1}{2}(\beta_1/\beta_2)^{\frac{1}{2}}\eta' e^{\psi(\eta')/2} \quad (a),\\ \xi'^2\frac{d\theta}{d\xi}\bigg|_{\xi=\xi'}&=\tfrac{1}{8}(\beta_1/\beta_2)^{3/2}\eta'^2\frac{d\psi}{d\eta}\bigg|_{\eta=\eta'}e^{\psi(\eta')/2} \quad (b).\end{aligned}\right\} \tag{2.17}$$

expressing the requirement of continuity of the physical variable at the surface of demarcation of both phases. This equation of fit (2.17) may be solved only approximately.

For this purpose we will determine the general character of the solution of $\psi(\eta)$. With this object we will estimate the value of $\frac{d^2\psi}{d\eta^2}$. From (2.12) $\psi(\eta) = ln\frac{\rho}{\rho_c}$, it follows: $\frac{d^2\psi}{d\eta^2} \propto \frac{d}{dr}\left(\frac{d}{dr} ln\rho/\rho_c\right)$. As the core is an aggregate of matter with low potential of energy, the density in the core will change but little from the centre to the interface. Therefore in the first approximation we can consider the gradient $ln\rho/\rho_c$ constant. Consequently we can deem with a certain degree of accuracy that the solution is determined by the equation

$$\frac{2}{\eta} d\psi/d\eta = -e^{\psi}, \tag{2.18}$$

of which the solution consistent with the initial conditions (2.13) is

$$e^{\psi} = 1/1 + \tfrac{1}{4}\eta^2. \tag{2.19}$$

Further, according to (2.12) and (2.19),

$$e^{\psi(\eta')} = \rho'/\rho_c = 1/1 + \tfrac{1}{4}\eta'^2, \tag{2.20}$$

whence

$$\eta' = 2\left(\frac{\rho_c}{\rho'} - 1\right)^{1/2} = 2\left(\frac{\rho_c}{\rho'}\right)^{1/2}\left(1 - \frac{1}{2}\frac{\rho'}{\rho_c} - \frac{1}{8}\left(\frac{\rho'}{\rho_c}\right)^2 - \ldots\right).$$

If the density does not decrease inwards, then $\rho' < \rho_c$ and for the first approximated solution of (2.17) we may take for the η' the following expression:

$$\eta' = 2(\rho_c/\rho')^{1/2}. \tag{2.21}$$

Therefore, taking into account (2.19), (2.20) and (2.21) it is easy to show that (2.17) becomes

$$\xi'\theta(\xi') = (\beta_1/\beta_2)^{1/2}\ (a), \qquad \xi'^2 \frac{d\theta}{d\xi}\bigg|_{\xi=\xi'} = -\tfrac{1}{2}(\beta_1/\beta_2)^{3/2}\ (b). \tag{2.22}$$

In this expression $\theta(\xi)$ is the solution of (2.4) with the initial conditions (2.7), *i.e.* $\theta = \theta(\xi, \omega_3)$, therefore ξ' in (2.22) will depend upon ω_3, *i.e.* $k_1 L$ and M. Thus (2.16) is a fundamental relation between the radius, the mass and the luminosity in the two-phase configuration considered, when ω_3 is known. For each type of solution in the envelope and for all values of β_1/β_2, we may find a value of ξ' such that for $\xi = \xi'$, $\theta(\xi')\xi' = (\beta_1/\beta_2)^{1/2}$, *i.e.* (2.22) (*a*) is satisfied. For this purpose it is necessary to find the point of intersection of the curve $\theta(\xi) = 1/\xi(\beta_1/\beta_2)^{1/2}$ with each curve of solution $\theta(\xi, \omega_3)$ passing through the point $\xi = \xi_1$, and possessing a given value of $\left|\frac{d\theta}{d\xi}\right|_{\xi=\xi_1}$ or ω_3.

On fig. 1, borrowed from N. Fairclough's paper,* are replotted the loci

* *M.N.*, **91**, 55, 1930.

of constant values $\xi_1\theta(\xi)\,.\,\xi/\xi_1=(\beta_1/\beta_2)^{1/2}$ (dotted curves) in the ξ, $\theta(\xi)$ plane, where each continuous curve is the solution of (2.4) vanishing at $\xi=\xi_1=1$

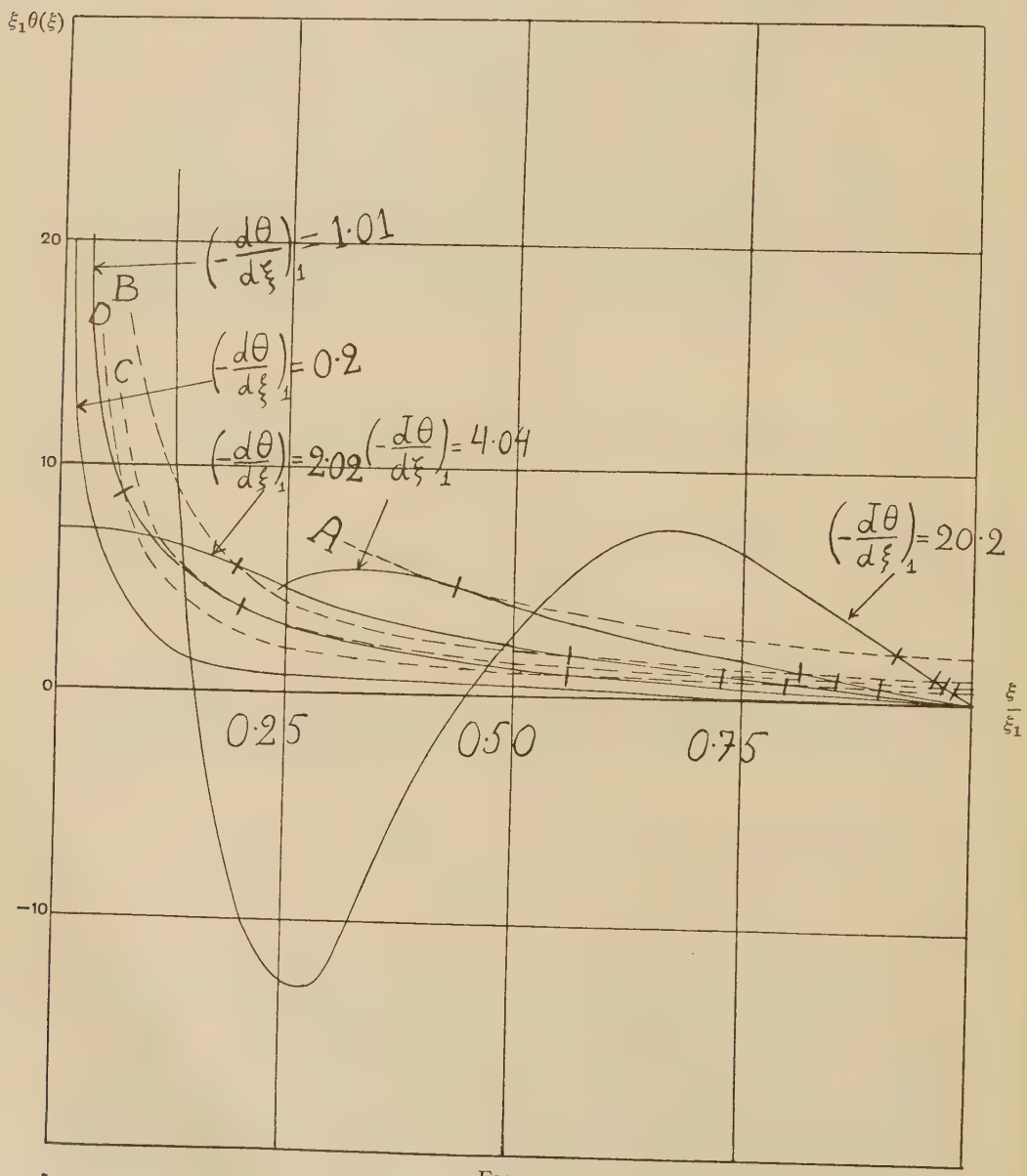

FIG. 1.

and possessing the given slope $\left|\frac{d\theta}{d\xi}\right|_{\xi=1}=\omega_3$. The curves $\xi\theta(\xi)=(\beta_1/\beta_2)^{1/2}$ are replotted for the following values of β_1 (β_2 is taken as constant for the model under consideration): $4\beta_2$ (the A-curve), β_2 (B-curve), $\frac{1}{2}\beta_2$ (C-curve) and $\frac{1}{4}\beta_2$ (D-curve).

First it is to be noted that for the values $\beta_1 > \beta_2$ all configurations are in general unstable, and the configurations with $\beta_1 > \beta_2$ cannot exist without introducing an artificial supporting surface at the interface.

For the configurations for which $\beta_1 \leqslant \beta_2$ the following deductions may be obtained from the discussion of fig. 1.

Case I.—If $\omega_3 > 2{,}018 \equiv \omega_3^\circ$ (collapsed solutions in the envelope, ω_3° corresponding to Emden's solution in the envelope), then there exist for each $\beta_1/\beta_2 \leqslant 1$ a point of intersection ξ', so that $\frac{d\theta}{d\xi} < 0$ in $\xi' \leqslant \xi \leqslant \xi_1$, and this number ξ' increases, when ω_3 increases (for β_1/β_2 fixed). (See the behaviour of ω_3 – curves in the region of large values of ξ'/ξ_1 in Milne's diagram.*) The corresponding configurations are collapsed.

Case II.—If $\omega_3 = \omega_3^\circ \pm h \equiv 2{\cdot}018 \pm h$ (Emden's and approximately Emden's solutions), then there exist for each $\beta_1/\beta_2 \leqslant 1$ two points of intersection ξ', such that $\frac{d\theta}{d\xi} < 0$ in $\xi' \leqslant \xi \leqslant \xi_1$; if $\omega_3 \to \omega_3^\circ + 0$ then we get $\xi_1' > \xi_2'$, which number ξ_2' is the one obtained as $\omega_3 \to \omega_3^\circ - 0$.

Case III.—If $\omega_3 < \omega_3^\circ$ (centrally condensed solutions), we may take into consideration three cases.

1. There may be a value of $\beta_1/\beta_2 \leqslant 1$, such that no point of intersection exists, except $\xi' = 0$ (for example, $\beta_1/\beta_2 = 1$ and the curve $\omega_3 = 1{\cdot}01$). 2. There may be a value $\beta_1/\beta_2 \leqslant 1$, such that there exists one and only one value of $\xi' \neq 0$, consistent with $\frac{d\theta}{d\xi} < 0$ in $\xi' \leqslant \xi \leqslant \xi_1$ (for example, $\beta_1/\beta_2 = \frac{1}{2}$ and the curve $\omega_3 = 1{\cdot}01$). 3. There may be a value $\beta_1/\beta_2 \leqslant 1$, such that there exist two and only two different values of $\xi' \neq 0$, consistent with $\frac{d\theta}{d\xi} < 0$ in $\xi' \leqslant \xi \leqslant \xi_1$ (for example, $\beta_1/\beta_2 = \frac{1}{4}$ and the curve $\omega_3 = 1{\cdot}01$).

Fig. 2 represents schematically the results of this investigation; in the $(\xi'/\xi_1, \beta_1/\beta_2)$-plane we replot the "$\omega_3 = \text{const.}$" curves. In the region of the unstable configurations $\beta_1/\beta_2 > 1$ we represent the ω_3-curves by a dotted curve. The dotted curves in the region $\beta_1/\beta_2 \leqslant 1$ are replotted approximately. As is easy to see, this diagram is analogous to Milne's diagram.† It ought to be noted that there exists a value $\omega_3 = \varpi_3$ (dotted curve in the $\beta_1/\beta_2 \leqslant 1$ region), so that all the curves $\omega_3 = \text{const.} > \varpi_3$ pass through the region $\beta_1/\beta_2 > 1$, and all the curves $\omega_3 < \varpi_3$ lie wholly in the $\beta_1/\beta_2 \leqslant 1$ region; therefore all the configurations with $\omega_3 > \varpi_3$, in general, may be unstable for them, as soon as $\frac{\beta_1}{\beta_2} \to 1 - 0$.

It is necessary to note that at the point $\xi = \xi'$ the condition (2.22) (*b*) ought also to be satisfied for the solutions $\theta(\xi, \omega_3)$; therefore, at the surface of demarcation $\xi = \xi'$ there may be a discontinuity of the slope of the order $-\xi^2\left[\frac{d\theta}{d\xi}\right]_{\xi=\xi'} + \frac{1}{2}(\beta_1/\beta_2)^{3/2}$. It is obvious that the value of this discontinuity depends upon ω_3, *i.e.* upon $k_1 L$ and M—this fact has first been noticed by

* *M.N.*, **92**, 628, 1932. † *Ibid.*

Chandrasekhar.* Therefore it is necessary to assume the existence of an intermediate region of incipient degeneracy, in which the slope $\frac{d\theta}{d\xi}(\xi, \omega_3)$ varies continuously, approaching its limiting value $-\frac{1}{2}(\beta_1/\beta_2)^{3/2}\,1/\xi'^2$.

In concluding we will determine the physical characteristics of our

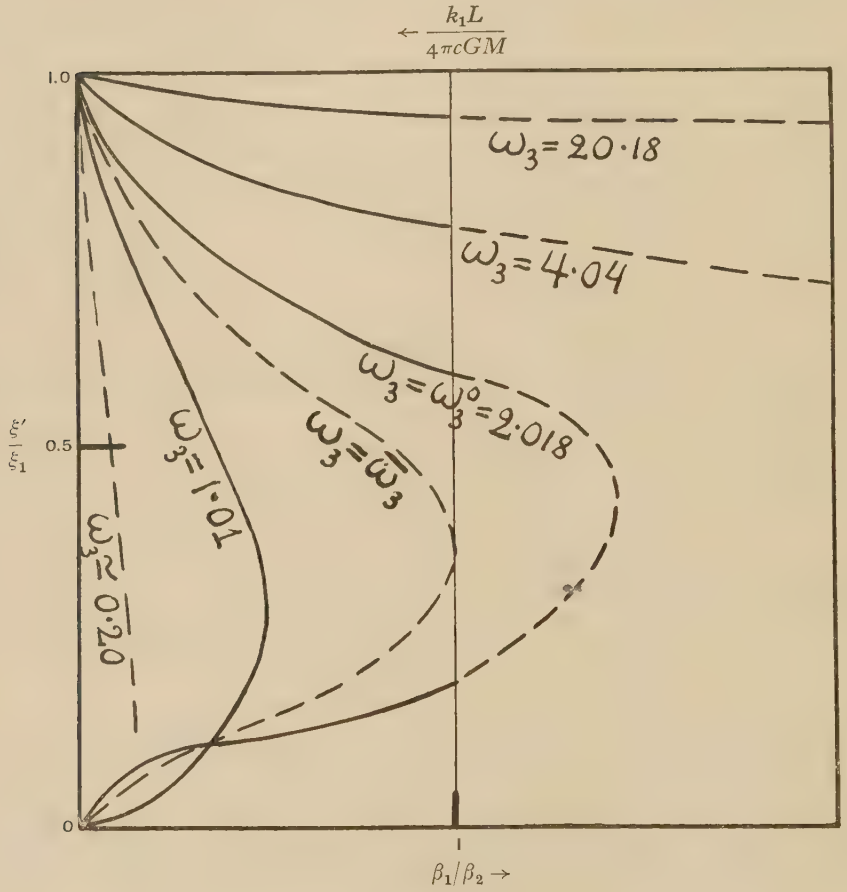

FIG. 2.

configurations and consider the general character of the variation of r_1 for different values of ξ'/ξ_1 and β_1.

Radius of the configuration:

$$r_1 = \frac{(R/\mu)^2}{K_3}\,\frac{1}{(\frac{1}{3}a\pi G)^{1/2}}\left(\frac{1-\beta_1}{\beta_1}\right)^{1/2}\frac{1}{\beta_2^{1/2}}\,\frac{\xi_1}{\xi'}. \tag{2.23}$$

Radius of the core:

$$r' = \frac{(R/\mu)^2}{K_3}\,\frac{1}{(\frac{1}{3}a\pi G)^{1/2}}\left(\frac{1-\beta_1}{\beta_1}\right)^{1/2}\frac{1}{\beta_2^{1/2}}. \tag{2.24}$$

* *Zs. f. Astroph.*, **5**, 321, 1932.

Mass of the core:

$$M(r') = \frac{2}{(\frac{1}{3}a\pi G^3)^{1/2}}\left(\frac{R}{\mu}\right)^2\left(\frac{1-\beta_1}{\beta_1}\right)^{1/2}\frac{1}{\beta_2^{3/2}}. \tag{2.25}$$

Mean density of the configuration:

$$\rho_m = \frac{3}{4\pi}\frac{K_3^3}{(R/\mu)^6}(\tfrac{1}{3}a\pi G)^{3/2}\left(\frac{\beta_1}{1-\beta_1}\right)^{3/2}\beta_2^{3/2}\left(\frac{\xi'}{\xi_1}\right)^3. \tag{2.26}$$

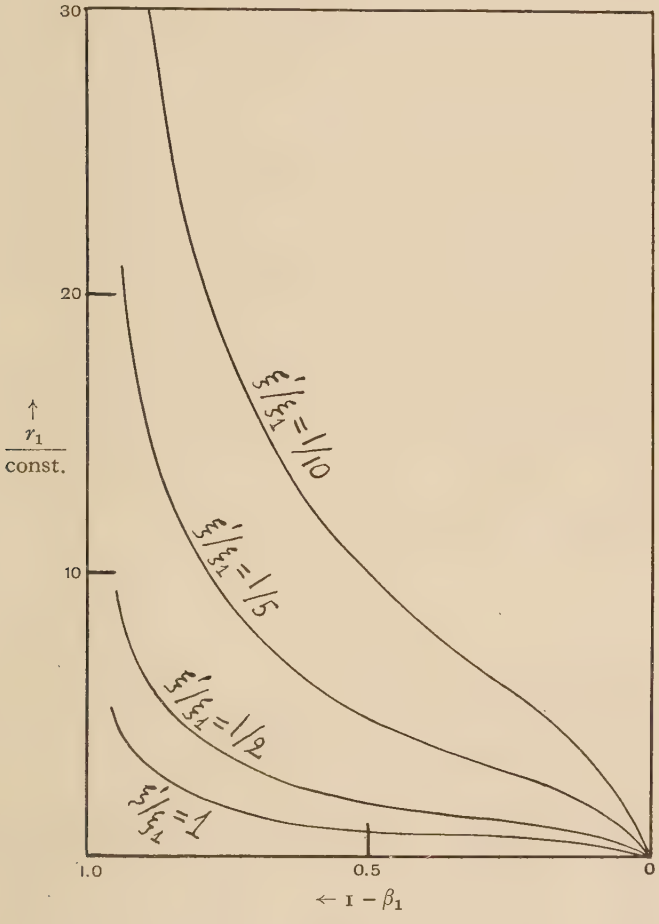

FIG. 3.

Mean density of the core:

$$\rho_m\,(\text{core}) = \frac{3}{2\pi}\frac{K_3^3}{(R/\mu)^4}(\tfrac{1}{3}a\pi)\frac{\beta_1}{1-\beta_1}. \tag{2.27}$$

Effective temperature:

$$T_e = \left(\frac{L}{ac}\right)^{1/4}\frac{K_3^{1/2}}{R/\mu}(\tfrac{1}{3}a\pi G)^{1/4}\left(\frac{\beta_1}{1-\beta_1}\right)^{1/4}\beta_2^{1/4}\left(\frac{\xi'}{\xi_1}\right)^{1/2}. \tag{2.28}$$

On fig. 3 in the $(\beta_1, r_1/\text{const.})$-plane are replotted the loci of the constant

relative core-radius ξ'/ξ_1. From the discussion of this diagram the following conclusions may be drawn. All configurations with large $(1-\beta_1)$ have a large external radius (for every value of relative core-radius ξ'/ξ_1), and all configurations with small $(1-\beta_1)$ have a small external radius (for every value of ξ'/ξ_1). This fact may correspond to the division of the stars of Nature into giants and dwarfs in general. For a fixed value of $(1-\beta_1)$ the external radius increases when the relative core radius decreases. This fact may, on the one hand (when $(1-\beta_1)$ is large), correspond to the division of giant stars into diffuse giants and massive giants of O-B type, and on the other hand (when $(1-\beta_1)$ is small), it may point to the division of dwarf-stars into ordinary dwarfs and white dwarfs. In both cases we see from (2.28) that the effective temperature increases, when ξ'/ξ_1 increases. This conclusion also agrees with Milne's results.*

Without pausing for a detailed analysis of the properties of the configurations examined in the present article, which is to be the object of a special discussion, the author is inclined to believe that this preliminary simple analysis of the equations of the problem permits a general inference as to the applicability of Milne's ideas in these directions.

In concluding the author expresses his gratitude to Prof. E. A. Milne, who kindly gave him his suggestions on the treatment of the problem, and to Dr. V. A. Ambarzumian and Dr. M. A. Leontovich, who offered some remarks of a physical character.

Academy of Science, U.S.S.R.,
Moscow :
1935 *December* 6.

THE VARIABLE SPECTRUM OF ζ TAURI.

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Observations relative to the spectrum of ζ Tauri (H.D. 37202 R.A. $5^h\ 31^m.7$, Dec. $+21°\ 5'$, Epoch 1900·0, Pm. Mag. 3·0, Sp. Type B3emnβ) have been previously communicated by me from photographs taken at the Norman Lockyer Observatory.†

Those photographs clearly indicated distinct changes in the appearance of $H\beta$ due either to the variable intensity of the central absorption or to the variable intensities of the bright components which lie on each side of it. In the last-mentioned reference I pointed out (p. 689) that "the line [K] seems also to be variable in harmony with the changes of $H\beta$."

Since the year 1928 only two publications can be found referring specially to the spectrum of this star. The first is by Miss Hazel Marie Losh ‡ in

* *Zs. f. Astroph.*, **5**, 321, 1932.
† *M.N.*, **83**, 328, 1923 ; **84**, 417, 1924 ; **88**, 689, 1928.
‡ *Pub. Obs. Univ. Michigan*, **4**, 1, 1931.

1931, and the second by Dean B. McLaughlin * in 1932. McLaughlin's reference is so short that it may be quoted here in full. He writes :

"A study of the spectrum of this star has been carried out by Miss Losh. Conspicuous change, evidently of long period, has occurred in V/R. In 1914 the violet component of each emission line was much the stronger. In 1921 and later years the components were equal. Unpublished data by Miss Losh show a marked fading of the emission after equality was reached. In 1831 the $H\beta$ emission was hardly as intense as was $H\delta$ in 1914 and 1921. Enhanced iron emission, present in 1914, is invisible now, but the central absorption lines remain visible."

Miss Losh's research is chiefly confined to measures of radial velocity, although she gives a list of the wave-lengths of the lines in the spectrum compared with the list I published in 1923.†

Miss Losh does not refer to any change of intensity of any line synchronising with the period of 133 days, and she specially states that the bright lines of $H\beta$ do not suffer any alteration, for she says (p. 3) that "during the observations of any one season the $H\beta$ emission has not been observed to change, indicating that its period of variation is long."

Coming now to my own observations made at Sidmouth, it may first be stated that all the negatives were taken with one instrument, namely, the Kensington prismatic camera, which has an objective of 9 inches aperture, 120 inches focal length, with a 9-inch prism of 45° angle, giving a dispersion of 46·0 mm. between $H\beta$ and $H\delta$.

The observations extend over a period of a little over twelve years, extending from 1923 January 11 to 1935 March 20, and consist of a series of 186 photographs.

The general spectrum may be briefly described as follows (see Plate 3) :—

The Balmer series of hydrogen lines are extremely sharp absorption lines usually photographed down to $H\epsilon$, but $H\beta$ is sometimes a single sharp absorption line, and sometimes a sharp absorption line bounded on each side by a comparatively broad emission line. This is the only hydrogen line that has bright components recorded here. The next strongest set of absorption lines is that of helium, the strongest being λ 4471, all of which are of a nebulous character.

Then follow in intensity the ionised metallic absorption lines, similar to those in α Cygni, namely, *Ca*, *Mg*, *Fe*, *Ti*, etc. Sometimes some of these lines are bounded on each side by an emission line.

A first general survey of the whole series of negatives leads one to the conclusion that not only is $H\beta$ variable, in that the strength of the central absorption varies and that the bright components changed their relative intensities, but that many of the other lines, particularly [K], alter considerably in intensity.

This investigation takes the form of the determination of the variations of the following :—

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* *Pub. Obs. Univ. Michigan*, **4**, 175, 1932.

† *M.N.*, **83**, 329, 1923.

(1) The intensity of the central absorption of $H\beta$.
(2) The intensity of some of the ionised lines.
(3) The combined intensity of the bright components of $H\beta$.
(4) The relative intensity of the bright components of $H\beta$.

While it was possible to make eye estimates or to measure with the wedge micrometer the difference of intensity between the bright components of the hydrogen line at $H\beta$, with fair accuracy, such was not the case with the other lines in the spectrum, because, owing to differences of exposure, different plate emulsions, quality of the night, etc., the intensity of the same lines from one negative to another might alter. Such an investigation requires for great accuracy spectro-photometric observations with the accompanying calibration technique. Fortunately the Sidmouth negatives include a whole series of plates of α Cygni, a star the lines in the spectrum of which are known not to suffer any change in intensity, covering about the same period of time as ζ Tauri. It was interesting, therefore, to make a careful examination of this set of negatives to investigate whether any changes in the intensity of the lines could be observed owing to the variability of instrumental, photographic, etc. conditions similar to those experienced in the case of ζ Tauri.

None, however, were detected, indicating that these sources of error were not sufficiently pronounced to make any marked changes.

Nevertheless the changes in the intensities of some of the lines in ζ Tauri were so prominent, in spite of the above-mentioned possible sources of error, that an attempt was made to determine them, even if only approximately.

This was done by selecting two negatives which represented extreme cases, *i.e.* when the intensities of the lines other than hydrogen (except $H\beta$) were at a maximum and a minimum and comparing all the other negatives with those two standards, using an arbitrary scale of differences.

The lines selected for intensity determination by this method were as follows :—

The Absorption	line	of *Ca*II	λ 3933·8.
,,	lines	*Si*II	λλ 4128·1, 4131·1.
,,	,,	*Fe*II	λλ 4173·5, 4178·9.
,,	line	*Fe*II	λ 4233·2.
,,	,,	*Fe*II	λ 4351·9.
,,	,,	*He*	λ 4471·7
,,	,,	$H\beta$	λ 4861·5.

To act as a check on any variations found in the eye estimates the absorption lines of [K] and $H\beta$ were measured with the wedge micrometer in the usual way. The line [K] was measured against $H\epsilon$ and $H\zeta$ so that two series of values were obtained for the variation of [K]. As [K] (λ 3968·4) lies close to $H\epsilon$ (λ 3970·1), which is assumed to vary with [K] and would no doubt influence the value of the intensity of $H\epsilon$, only the differences between $H\zeta$ and [K] are referred to here. As a matter of fact the changes of [K] in relation to both $H\epsilon$ and $H\zeta$ exhibited very similar variations.

For the determination of the intensity of the two bright components

W. J. S. Lockyer, The Variable Spectrum of ζ Tauri.

of $H\beta$, as a whole and not individually, the following scale was adopted. Each negative was considered by itself and the intensity estimated on a scale from 0 (absent) to 6 (very strong).

Not all the data have been plotted in fig. 1, but only those referring to the absorption lines of $H\beta$, [K], *Si*II and the bright lines of $H\beta$. The points are plotted under each month, J standing for January. The filled-in circles represent data from good, and the open circles from less accurate measures. Each circle represents the mean of all the negatives taken in any one month.

The highest points of all the curves represent strong absorption and the lowest points weak absorption. On the other hand, in the case of the bright lines of $H\beta$ it should be noticed that the *scale has been reversed*. This was done because the bright lines behave *inversely* to the absorption lines, *i.e.* when the $H\beta$ absorption line is strong the bright components are weak. Coming now to the discussion of the data it was apparent at once that there was a long period of variation present. In each case, therefore, a smooth mean curve was drawn to satisfy the groups of observations as far as possible.

For the most part the curves, whether exhibiting absorption or emission, or whether measured with the wedge micrometer or by eye estimation, all show similar tendencies. They all display a fall from 1923 to a minimum about the end of 1925, a fairly rapid rise to about the middle or end of 1928, then generally a slow fall up to the present date (1935 March). The curves for the eye estimates of [K] and *Si*II do not, however, maintain this fall after the middle of the year 1930, but display two subsidiary maxima. It will be noticed that the curve from the wedge measures of [K] does not exhibit those subsidiary waves but falls like the others. A sufficient number of years is not available to determine the period of variation.

After the smooth curves had been drawn to represent the long period variation there seemed distinct evidence for the existence of a shorter period of the order of about 430 days. Synchronous and similar waves seemed to be inherent in all the curves, whether by eye estimate or wedge measurement, adding weight to their existence. Their confirmation must be left to other observers, employing more accurate methods of measurement, before their reality can be established.

Reference has been previously made to McLaughlin's and Miss Losh's remarks on the variations of the relative intensity of the bright components of $H\beta$, and a very close examination of their appearance and relative intensities has been specially made from the Sidmouth negatives.

Each of these bright lines varies in width from very broad lines always contiguous to each side of the central absorption, to exceedingly thin sharp lines sometimes in contact with and sometimes detached from the central absorption line.

During some months, especially the latter end of the year 1927 to the beginning of 1929, the combined intensity of those lines is zero or the components have vanished. There is no doubt about these being gradual variations due to the bright components of $H\beta$ becoming strong and weak, but whether there are *regular* changes in the relative intensities of the violet

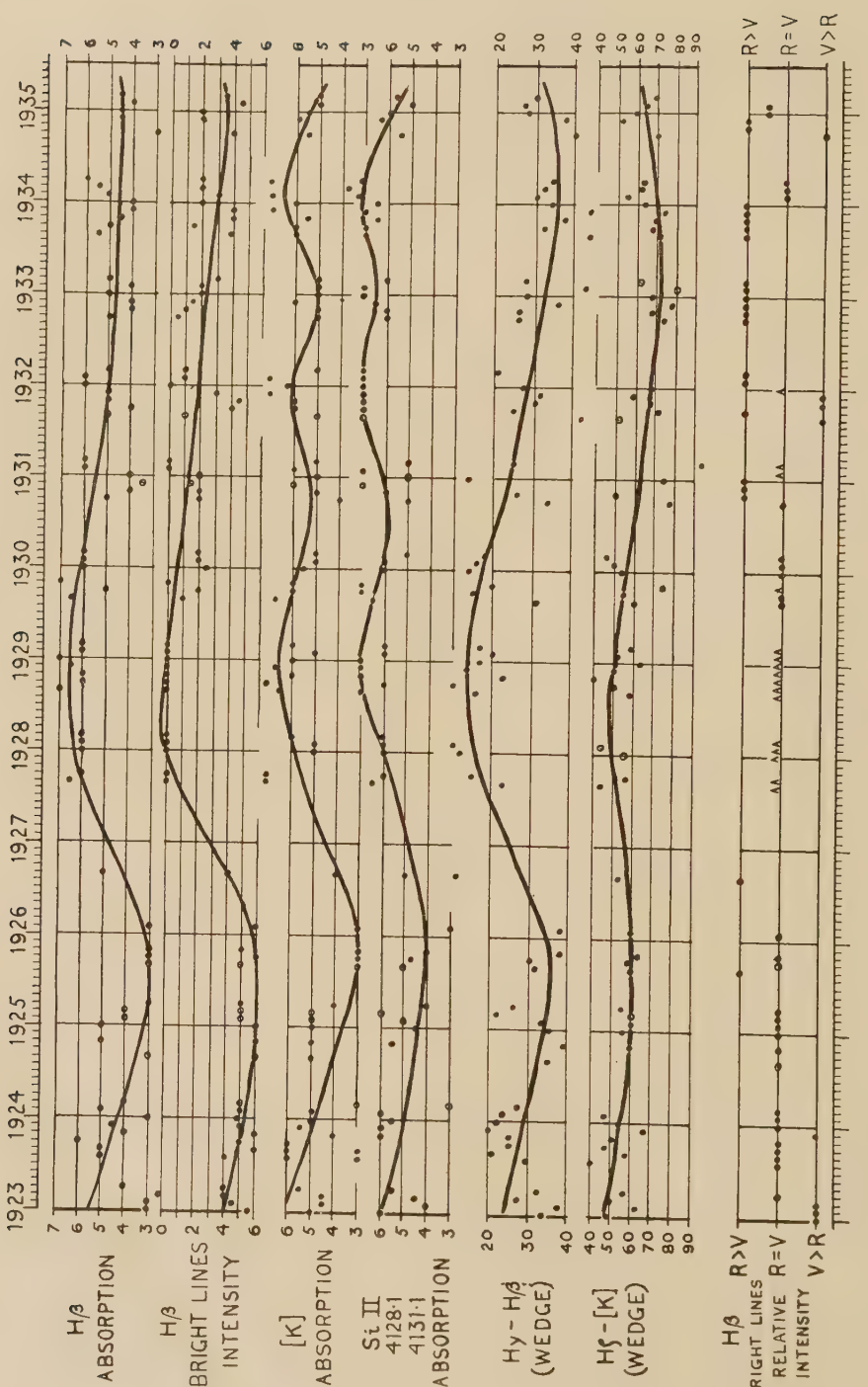

1923
1924
1925
1926
1927
1928
1929
1930
1931
1932
1933
1934
1935
Hβ ABSORPTION
Hβ BRIGHT LINES INTENSITY
[K] ABSORPTION
Si II 4128·1 4131·1 ABSORPTION
Hγ - Hβ (WEDGE)
Hζ - [K] (WEDGE)
Hβ BRIGHT LINES RELATIVE INTENSITY
R>V
R=V
V>R

FIG. 1.

and red components of this line, the length of time over which observations are available is not sufficient to determine this.

The observations made at Sidmouth and plotted at the bottom of fig. 1 indicate that from 1923 April to 1926 February, when both bright components were strong, they were of equal intensity. From 1927 September to 1929 March they were both absent (marked A in fig. 1). From 1929 to 1930 October, when they were both weak, they were of equal intensity again. From that date up to the present time (1935 March) the red component has for the most part been a little stronger than the violet.

The following statement gives a general and interesting summary of all the available information published by other observers, together with the data from the Sidmouth observations :—

1896 Feb.	No bright lines	Dr. McClean's negative.
1912 Nov.	$V >> R$	Merrill and Burrell, *Ap. J.*, **78**, 103.
1914	$V >> R$	Losh, *Pub. Obs. Univ. Michigan*, **4**, 3, 1931.
1915 Jan.	$V >> R$	Lockyer, Sidmouth Photograph.
1921	$R > V$ (*L*)	Losh, *Pub. Obs. Univ. Michigan*, **4**, 3, 1931.
1923–1924	$V = R$	,, ,, ,,
1923 Apr.–1926 Feb.	$V = R$	Lockyer, Sidmouth Photographs.
1925–1927	$R > V$ (*L*)	Losh, *Pub. Obs. Univ. Michigan*, **4**, 3, 1931.
1927 Sept.–1929 Mar.	No bright lines	Lockyer, Sidmouth Photographs.
1929 Sept.–1930 Oct.	$V = R$	,, ,, ,,
1931 Jan.–1935 Mar.	$R > V$	,, ,, ,,

With regard to the year 1921, Miss Losh states, "the tendency was for the red component to be slightly stronger than the violet," and for the years 1925 to 1927, "they were still nearly equal in 1925 to 1927, but on the whole the red component seemed stronger than the violet." Coupling these observations with those made at Sidmouth from 1923 April to 1926 February, it may be assumed that the components were about the same intensity for the whole period from 1921 to the beginning of 1927.

Thus since 1912 there seems to have been a gradual brightening up of the red component in relation to the violet, excluding of course the period when both components vanished, as they did, from 1927 September to 1929 March. It will be interesting to watch their future behaviour.

DESCRIPTION OF PLATE 3

This plate illustrates two spectra of ζ Tauri, No. (2) and No. (3), with comparison spectra of No. (1) α Cygni (Sp. Type A2), typical of ionised metallic lines, and No. (4) γ Orionis (Sp. Type B2), typical of helium lines. Spectrum No. (2) was taken in 1928 October 13, and shows the condition when $H\beta$ and the ionised metallic lines exhibited the greatest absorption and $H\beta$ did not display any bright components. Spectrum No. (3) was secured in 1925 October 11 when the absorption of $H\beta$ and the ionised lines were least and when the bright components of $H\beta$ were very strong.

The above photographs were all broadened by the Clepsydra apparatus previously described (*M.N.*, **85**, 47, 1924), but it should be noted that in spectrum No. (3) the absorption line at approximately λ 3912 is an artificial line.

THE NEBULAR SPECTRUM OF NOVA HERCULIS.

Arthur Beer, Ph.D.

1. *The Material*

The spectroscopic material of Nova Herculis 1934 obtained at the Solar Physics Observatory, and at the University Observatory, Cambridge, has been partly discussed, so far as the earlier nova stage is concerned, in previous papers.* For a few weeks in 1935 April–May, when the nova was fainter than 11^m, observations had to be discontinued. On May 18, after the star had become brighter again, the first plate of the new series was secured; it showed that since the middle of April the spectrum had totally changed towards the "nebular-stage." Since then 100 plates of the second series of spectra of the nova have been obtained with the 25-inch Newall refractor. The writer is very grateful to Professor Stratton who has kindly put this material at his disposal for the present investigation. He is also much obliged to Mr. H. E. Green, who kindly allowed him to use for the same purpose his ultra-violet plates secured in this period.

In the following Table I the first column gives the plate-number, the second the mid-time of exposure, the third its duration. In the fourth column: S = short 10-inch camera used; L = long 20-inch camera; R = red region, λ 5400 – $H\alpha$, obtained at the same time, using two parts of different sensitive plates in the same plate-holder, *i.e.* Ilford Astra II as usual for the blue, and Ilford Hypersensitive Panchromatic for the red region. The last column M gives the magnitude of the nova.

An asterisk in the first column of Table I denotes that the exposure was extended over more than one night.

Details of the ultra-violet plates, instrument, etc., will be found in Green's paper (*loc. cit.*). The plates were Ilford Hypersensitive Panchromatic. The following three plates were used :—

No. = C 32,	Date = 1935 July 9·05,	E = 150^{min},	M = $7^m.3$,
C 35,	July 21·04,	150	$7^m.2$,
C 38,	Aug. 23·02,	285	$6^m.7$.

The magnitude of the nova is taken from a mean light curve, shown in fig. 1. In plotting the observations nearly the same sources could be used as cited in the previous paper (*loc. cit.*). Again M. Beyer has kindly contributed unpublished data.†

* F. J. M. Stratton and A. Beer, *M.N.*, **95**, 432, 1935; A. Beer, *M.N.*, **95**, 538, 1935; E. G. Williams, *M.N.*, **95**, 573, 1935; H. E. Green, *M.N.*, **95**, 694, 1935.

† In *Harvard Circ.* 898, L. Campbell has published a very complete study of the light curve in the first nova period, taking carefully into consideration individual corrections. In a similar way, regarding the differences between the observers, most of the same material as was used by the present writer in *M.N.*, **95**, 538, 1935, is treated in valuable publications of the Vienna Observatory (K. Graff, H. Krumpholz, *Mitt. d. Wiener Sternw.*, No. 4, 1935; *Akad. d. Wiss. Wien*, IIa, **144**, H. 5–8, 1935). See also G. P. Kuiper, *P.A.S.P.*, **47**, 99, 165, 229, 270; 1935.

TABLE I

No.	Date	*E*	*C*	*M*	No.	Date	*E*	*C*	*M*
	1935	min.		m		1935	min.		m
T 445 *	May 18·48	500	*S R*	10·5	T 495	Aug. 9·90	30	*L*	7·0
446	,, 20·93	90	*S*	10·0	496	,, 9·95	80	*L*	7·0
447	,, 21·04	120	*S R*	10·0	497	,, 10·02	40	*L*	7·0
448 *	,, 22·87	190	*S*	9·7	498	,, 10·98	240	*L R*	6·9
449	,, 29·00	255	*S R*	8·7	499	,, 13·98	240	*L R*	6·9
450	,, 29·92	15	*S R*	8·6	500	,, 18·98	240	*S R*	6·7
451	June 2·98	110	*S R*	8·4	501	,, 21·04	300	*S R*	6·7
452	,, 3·99	235	*S R*	8·5	502	,, 21·96	300	*S R*	6·7
453	,, 8·99	210	*S R*	8·2	503	,, 22·96	300	*S*	6·7
454	,, 10·93	55	*S*	8·0	504	,, 25·92	180	*L*	6·6
455	,, 11·01	135	*S*	8·0	505	,, 27·90	135	*L*	6·6
456	,, 12·00	165	*S*	8·0	506	Sept. 2·93	210	*S R*	6·6
457	,, 12·94	90	*S R*	7·9	507	,, 4·98	30	*S*	6·6
458	,, 13·93	45	*S R*	7·9	508	,, 6·90	60	*S*	6·6
459	,, 14·99	120	*S*	7·9	509	,, 7·94	120	*S*	6·6
460	,, 17·01	150	*S*	7·8	510	,, 11·90	100	*S*	6·7
461	,, 21·99	220	*S*	7·8	511	,, 12·93	60	*S*	6·6
462	,, 22·94	40	*S*	7·7	512	,, 13·92	10	*S*	6·7
463	,, 23·01	190	*S*	7·7	513	,, 14·86	20	*S R*	6·7
464	,, 23·98	215	*S*	7·7	514	,, 15·88	160	*S R*	6·7
465	,, 25·00	230	*S*	7·6	515	,, 17·90	150	*L*	6·7
466 *	,, 27·73	380	*S R*	7·5	516	,, 19·85	30	*L*	6·8
467 *	,, 29·64	410	*S R*	7·5	517	,, 19·89	30	*L*	6·8
468 *	July 2·00	445	*S*	7·5	518	,, 20·93	80	*L*	6·8
469	,, 3·95	60	*S*	7·3	519 *	,, 23·99	170	*L*	6·7
470	,, 5·00	260	*S*	7·4	520	,, 25·94	340	*L*	6·7
471	,, 5·95	90	*S*	7·5	521	,, 29·82	120	*L*	6·7
472	,, 7·01	290	*S*	7·5	522 *	Oct. 3·09	835	*L*	6·7
473	,, 8·06	180	*S*	7·4	523	,, 6·80	50	*L*	6·6
474	,, 10·00	175	*S R*	7·3	524	,, 6·88	70	*L*	6·6
475	,, 13·04	20	*S*	7·1	525	,, 7·90	350	*S R*	6·6
476	,, 14·02	195	*S*	7·1	526	,, 8·94	250	*S*	6·6
477	,, 18·00	180	*S*	7·2	527	,, 10·88	335	*S*	6·7
478	,, 19·04	210	*S*	7·2	528	,, 11·96	210	*S R*	6·7
479	,, 21·02	210	*S R*	7·2	529 *	,, 14·56	480	*S*	6·7
480	,, 22·96	20	*L*	7·2	530 *	,, 20·65	420	*S R*	6·7
481	,, 23·98	135	*L*	7·2	531 *	,, 24·37	745	*S*	6·6
482	,, 25·00	190	*L*	7·2	532	,, 29·83	60	*S*	6·8
483	,, 27·92	20	*S*	7·1	533 *	Nov. 3·68	530	*S*	6·8
484	,, 31·03	255	*S R*	7·0	534	,, 9·77	120	*S*	6·6
485	Aug. 1·95	130	*L*	7·0	535	,, 11·73	30	*S*	6·6
486	,, 4·90	40	*L*	7·0	536	,, 12·75	35	*S*	6·6
487	,, 4·96	40	*L*	7·0	537	,, 13·78	50	*S*	6·6
488	,, 5·89	35	*L*	7·0	538	,, 23·74	40	*S*	6·8
489	,, 5·93	35	*L R*	7·0	539 *	,, 30·51	255	*S*	6·8
490	,, 5·96	15	*L*	7·0	540	Dec. 2·72	50	*S*	6·8
491	,, 5·98	15	*L*	7·0	541 *	,, 4·77	450	*S*	6·8
492	,, 6·03	40	*L*	7·0	542	,, 15·79	70	*S*	6·7
493	,, 7·94	55	*L*	7·0	543	,, 16·78	170	*S*	6·7
T 494	,, 8·00	30	*L*	7·0	T 544 *	Dec. 21·44	240	*S*	6·8

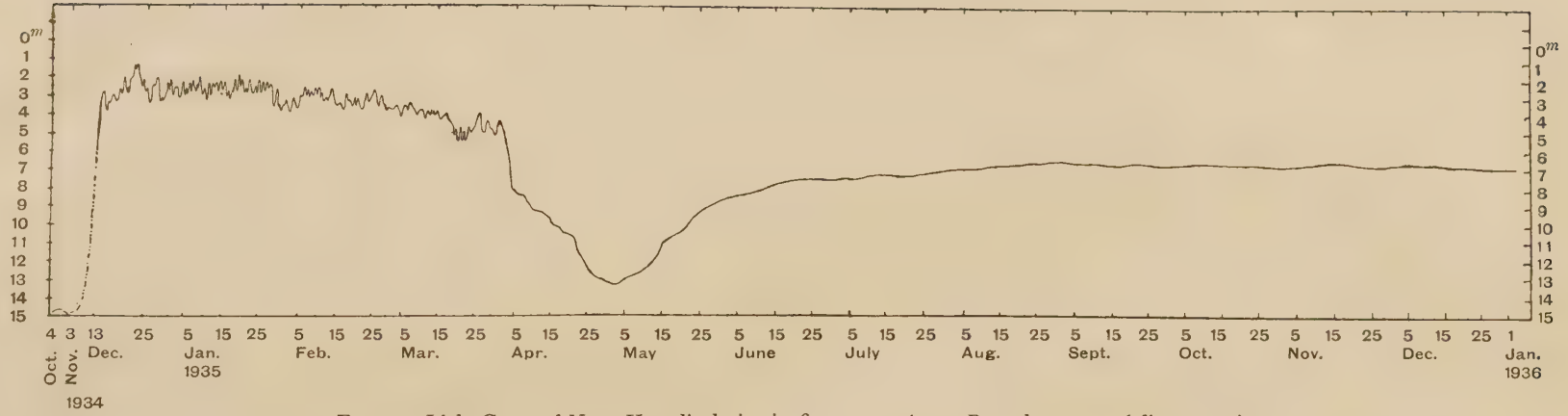


FIG. 1.—*Light Curve of Nova Herculis during its first year.* (1934 *December* 13–1936 *January* 3.)

The diagram illustrates the steep increase in brightness, from $13^{m}.2$ to $8^{m}.5$, which the nova experienced during May, whilst in June it gained only one magnitude, reaching at the end $7^{m}.5$; and after gaining $0^{m}.5$ in July, and about another $0^{m}.5$ in August, we note now for more than four months since then a very steady brightness, with only small and slow variations between $6^{m}.6$ and $7^{m}.0$.

2. *The Measurements*

For the reduction the selected T-plates were measured as usual four times, twice in both directions, with the Zeiss-comparator. The iron spark served as comparison spectrum, except for T 466, where the Balmer-lines of ζ Lyræ were used.*

The accuracy of the results was better than was to be expected in view of the broad and hazy character of most of the lines at this spectral stage. All available details of the structure of the spectrum were measured, especially the edges of the bright bands, the positions of maxima, different components, and all dark gaps or lines.

3. *The Structure of the Bright Bands*

In view of the difficulty of identifying broad bright unsymmetrical bands of varying width a careful study was made of the structure of the brightest bands on some short exposure plates taken with the long camera; such plates were secured from time to time to avoid the usual over-exposure of the main bands, *i.e.* especially of *Hγ*, *Hβ* and of [OIII] λ 4363, 4959, 5007. The most striking feature of the bright bands is a strong maximum near the end of shorter wave-length. A short description of some details of these plates may be of interest.

Plate 482.—*Hγ*, extending from λ 4333·6 to 4345·2, shows two narrow maxima, one at 4336·4 and a fainter one at 4338·2. A gap appears between 4341·3 and 4342·0. A faint emission is centred at 4331·5. *Hβ* shows two components, the one 4955·4–4958·4, the other 4959·2–4961·8; further, a faint emission at 4965·3. λ 4363 has two components, 4356·3–63·1, 4363·8–67·4. λ 4959 has its violet component centred at 4954·6, and its red at 4958·6, the latter extending from 4957·1 to 4960·1. λ 5007 has the one component centred at 5001·5, the red at 5005·3, a third faint at 5010·0.

Plate 485.—λ 4363 shows a stronger component from 4357·2 to 4363·3, with a gap at 4360·3, and a fainter emission from 4365·2 to 4366·3. λ 4959 has two maxima at 4954·6 and 4959·2. λ 5007 has a violet component 5000·0–5003·0, and a red one 5004·3–5006·6, and two faint maxima at 4996·2 and 5010·2.

Plate 487.—*Hγ* has a strong component from 4335·0 to 4340·1, and another 4342·2–44·3. *Hβ* is strong between 4863·8 and 4865·0, a fainter emission appears between 4860·9 and 4862·9, and another at 4855·4–59·7.

* This was a long exposure plate on which it was hoped that there would be a background of continuous spectrum; ζ Lyræ replaced the iron comparison with a view to a possible temperature determination.

λ 4363 shows gaps at 4362·8, 64·1, 65·5, 67·5. λ 4959 has gaps at 4959·8, 62·9, 65·8. λ 5007 shows gaps at 5010·5 and 5011·8.

Plate 489.—*Hγ* has its strong part from 4334·6 to 4341·5. *Hβ* shows gaps at 4862·8 and 4864·9, and a maximum at 4866·0. λ 4363 has gaps at 4367·3 and 4369·5. λ 4959 has gaps at 4959·6 and 4962·0. λ 5007 has gaps at 4998·0, 99·8, 5004·1, 07·5, 10·5, 13·7.

Plate 498.—*Hγ* shows, besides a faint maximum at 4333·1, four emissions at 4334·9–37·2, 4339·5–40·6, 4342·6–43·5, 4344·1–45·1. *Hβ* has one strong component from 4854·4 to 4859·1, and a fainter one from 4859·8 to 4861·5. λ 4363 has its strong part from 4356·5 to 4368·4 with five knots between at 4358·4, 60·6, 62·6, 64·5, 65·8. λ 4959 shows three components between respectively 4947·7–50·4, 4952·3–58·7, 4960·1–65·6. λ 5007 exhibits four emission-knots at 5000·5, 02·9, 06·5, 09·3.

In trying to get a consistent picture of the general structure of the bands, based mainly on the details mentioned above, the data of these five plates were graphically combined. Expressed in km./sec., *i.e.* displacements from the normal position of the line, allowing for the reduction to the Sun, we may describe the general result as follows :—

Hβ : Four components with central displacements of −4·4, −0·5, +2·9, +4·3 A., or −272, −31, +179, +265 km./sec. respectively.
Hγ : Five components with −491, −297, −28, +173, +283 km./sec.
λ 4363 : Five components with −564, −323, −96, +124, +261.
λ 4959 : Five components with −599, −260, −18, +145, +308.
λ 5007 : Five components with −555, −312, −78, +156, +264.

We can see that the main features of the structure agree well for the lines in question. But it is clear that it will prove difficult to obtain a more uniform picture ; this may be in slight part due to the effect of blending with other lines in the neighbourhood. For instance, we may have to expect within λ 5007 the presence of the strongest components of λ 5005·1 *N*II (10), and in λ 4363 of 4366·9 *O*II (7), etc. Neglecting this, we may nevertheless combine all the data given above, and state the presence of five components with the respective displacements : I, −550 ; II, −290 ; III, −50 ; IV, +170 ; V, +280 km./sec. The mean widths *w* of the bands measured above are : *Hγ*, 14·6 ; *Hβ*, 15·2 ; λ 4363, 16·8 ; λ 4959, 21·2 ; λ 5007, 19·6 A. ; for all 17·5 A.

The mean of the displacements of their centres from the undisplaced positions comes out at −125 km./sec. This statement should suggest a correction to be applied to the measured band-positions in making the identifications : Strong lines, with $w = 18$ A., in which therefore probably all components are traced on the plate should be corrected by +125 km./sec., say by +2·0 A. Very weak lines with *w* about 5 or 6 A. may show the main component II only, with −290 km./sec., corresponding to −4·5 A. displacement. For medium lines with $w < 18$ A. we may assume as displacement −3·0 A. Lines with $w > 18$ A. are obviously blends, and no unambiguous conclusions can be drawn.

With regard to the structure of the composite strong radiation at λ 4640,

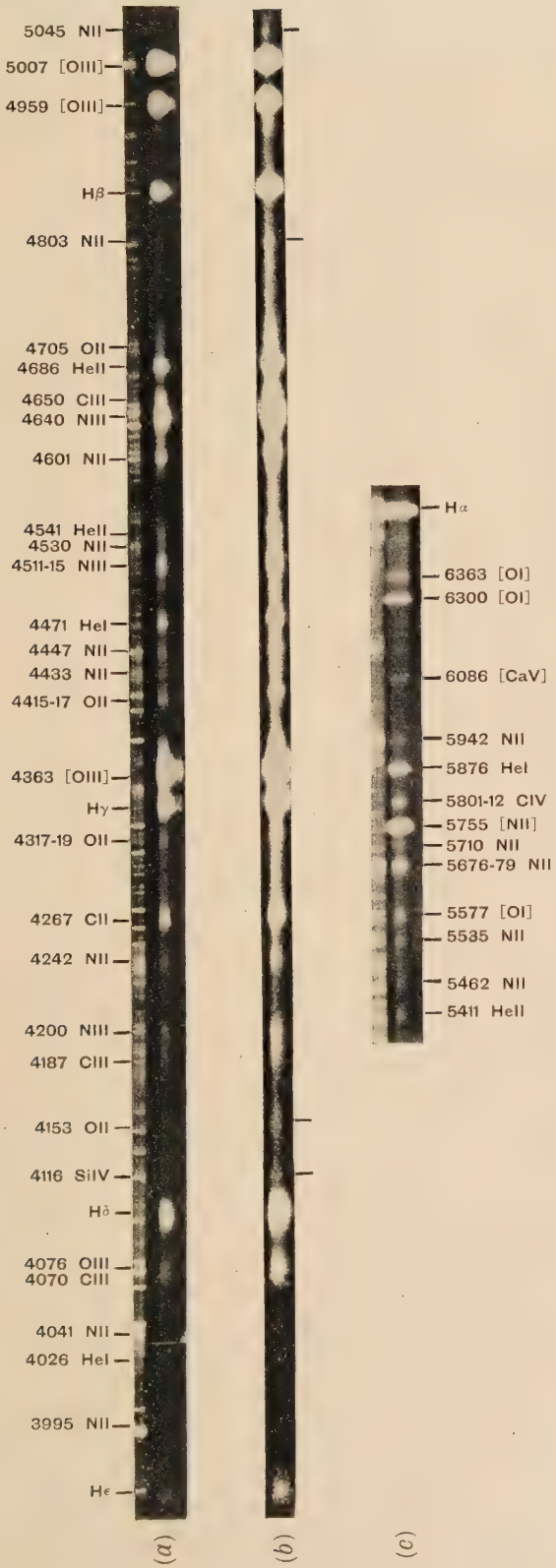


FIG. 2.—(*a*), *Plate* T530 (1935 *Oct.* 20.65; *top spectrum Fe-spark*); (*b*) *and* (*c*), *Plate* T467 (1935 *June* 29.64, *blue and red region*).

a nova-band which attracted considerable interest especially in the case of Nova Geminorum II,* our plate T482 shows the following details: A strong component extends from 4635·0 to 4637·3, two fainter ones show centres at respectively 4640·6 and 4643·6, one fainter still at 4638·9, and the faintest component, centred at 4632·6, goes from 4631·8 to 4633·3. T498 shows the strongest component with limits at 4634·1 and 4638·8. Applying to λ 4936·3, as the centre of this main component, the correction +4·5, corresponding to the displacement −290 km./sec. for component II, as already mentioned, we may hope to trace the origin of this strongest 4640-component; we obtain 4640·8, and find coincidence with λ 4640·6, *N*III (10). The other weaker maxima to be expected from this source should be at λ 4631·8, 4639·9, 4643·4, 4644·9. The observed maxima are at 4632·6, 4638·9, 4640·6 and 4643·6. Divergences may perhaps be accounted for by the presence of displaced maxima of 4641·8 *O*II (9), 4643·1 *N*II (8), 4647·4 *C*III (10). The width of the band on plates of long exposure, λ 4622–4662 A., suggests a group of origins lying between 4633 and 4655; the presence of 4634·2 *N*III (8), and 4650·2, 4651·4 *C*III (9, 8), 4649·1 *O*II (10), and 4658·6 *C*IV (5) is indicated.

4. *The Bright Band Spectrum*

To enable a proper identification, the centres of possible bands were derived from the edge-measures and special regard was paid to the band-widths as suggested in the last section. Unusual widths showed that two or more normal sources were blended into one band; weak narrow bands indicated the presence of strong displaced maxima only, etc. In identifying the lines some recent publications proved very useful, especially the works of B. Edlén and C. H. Payne-Gaposchkin.†

(a) *The Ultra-violet Spectrum* (λ 3120–*Hε*)

The reduction of the objective prism-plates showed especial difficulties, because it was here impossible to proceed step by step with the Hartmann-Cornu formula, as Green could do in reducing the nova-absorption-spectrum of 1935 February (*loc. cit.*). No certain nova lines were at our disposal at the extreme ultra-violet end of the spectrum to be used as standards, and every extrapolation with the formula failed on account of the long range to be bridged. Fortunately, on the same night as the nova plate C 35 a

* F. J. M. Stratton, *Annals Solar Phys. Obs., Cambridge*, **4**, part 1, 57, 1920; W. H. Wright, *Lick Publ.*, **14**, part 2, 84, 1926.

† C. E. Moore, Multiplet Table, Princeton, 1933; C. S. Beals, *D.A.O.*, **4**, 271, 1930, and *D.A.O.*, **6**, 95, 1934; H. Spencer Jones, *Ann. Cape*, **10**, part 9, 1931, and *M.N.*, **94**, 816, 1934; C. H. Payne-Gaposchkin, *Zeitschr. f. Astrophysik*, **7**, 1, 1933; B. Edlén, *Zeitschr. f. Astroph.*, **7**, 378, 1933; B. Edlén, *Z. f. Phys.*, **87**, 217, 1933; B. Edlén, *Nova Acta Reg. Soc. Scient. Uppsala*, ser. iv., **9**, No. 6, 1934; *Transactions I.A.U.*, **5**, 184, 1936. F. J. M. Stratton, *Handb. d. Astrophysik*, VI, 2, 279, etc., 1928, and vol. vii (part vi, 3) 1936; W. H. Wright, *Lick Publ.*, **14**, 1921–26; I. S. Bowen, *Ap. J.*, **81**, 1, 1935; R. H. Stoy, *P.A.S.P.*, **46**, 297 and 362, 1934; **47**, 162, 1935.

plate of α Cygni had been obtained too, which showed a good absorption spectrum in the ultra-violet. The lines in this plate could be easily identified, and as the α Cygni plate and the nova plate had the same scale, it was possible

Table II

No.	*C* 32	*C* 35	*C* 38	λ_m	Remarks
1	3115	3114	3118	3116	*v e*
2	3136	3135	3131	3134	*r e*
3		3153		3153	*v e u*
4		3171		3171	*r e u*
5		3228		3228	*n m vy u*
6	3252	3256	3252	3253	*v e u*
7	3270	3267	3270	3269	*r e u*
8		3291		3291	*v e*
9		3306		3306	*g u*
10		3317		3317	*r e*
11		3326		3326	*v e f ext*
12	3334	3334		3334	*r e f ext*
13	3335	3337	3333	3335	*v e*
14	3350	3346	3352	3349	*r e*
15		3356		3356	*v e u*
16		3364		3364	*r e u*
17	3374	3380	3376	3377	*v e*
18	3389	3390	3388	3389	*r e*
19	3406	3397	3402	3402	*v e u* ; on *C* 35 *c f ext*
20	3415		3413	3414	*r e*
21	3418	3422	3418	3419	*v e*
22	3434	3434	3436	3435	*r e*
23	3437	3436		3436	*v e* ; 23–25 : broken
24		3440	3440	3439	*g*
25	3453	3450	3453	3452	*r e*
26		3479		3479	*v e*
27		3501	3503	3502	*r e* = 28 *v e*
28			3514	3514	*r e u*
29		3533		3533	*v e u*
30		3565		3565	*r e u*
31		3654	3662	3658	*v e*
32	3705	3706	3685	3699	*r e, g u*
33		3720	3717	3718	*v e, g u*
34		3729	3734	3732	*r e, g u*
35	3738	3743		3740	*v e*
36	3749	3750	3754	3751	*v e* ; 36–37 : broken
37	3777	3770	3776	3774	*r e*
38	3793	3788		3790	*v e*
39	3819	3805		3812	*r e, vy u*
40	3828	3830	3832	3830	*v e*
41	3847	3848	3847	3847	*r e* ; on *C* 35 *f ext u* to 3856
42	3862	3865	3863	3863	*v e*
43	3882	3877	3882	3880	*r e*
44	3883	3886	3886	3885	*v e* ; on *C* 35 *m* at 3886–3894
45	3901	3898	3901	3900	*r e* ; 3894 *g u*

to choose on the latter three suitable standard lines : λ 3124·6 (=3115·7 *O*III, 3121·7 *O*III, 3132·9 *O*III 6); λ 3342·0 (=3340·9 *O*III, 3342·8 [*Ne*III]); λ 4101·8 (=*Hδ*).

The plates themselves, especially in the far ultra-violet, were difficult to measure, and, therefore, having regard to the different conditions of exposure, etc., the agreement between the three plates, which were used as being the best in focus, can be regarded as sufficient.

Table II gives in the fifth column, λ_m, the mean wave-lengths. The last column describes the appearance of the measured features with the following abbreviations : *c* = centre, *d* = dark, *e* = edge, *em* = emission, *ext* = extension, *f* = faint, *g* = gap, *l* = line, *m* = maximum, *n* = narrow, *p* = part, *r* = red, *s* = sharp, *u* = uncertain, *vy* = very, *v* = violet.*

The following Table III contains in the successive columns : The Nos. of the λ_m of Table II, which were combined to define each band ; its width *w* ; the position λ_c of its centre ; the intensity, *I* ; the presumed origin, *O* ; the laboratory wave-length, λ_L ; the multiplet description, *M*, separately listing the transition and the inner quantum numbers ; the laboratory-intensity, I_L, of the suggested identification.

The intensity estimates, *I*, are on an arbitrary scale and are simple eye estimates, made with the help of plate enlargements and microphotometer-tracings. Here may follow also some intensities for λ ⩾ *Hε* (from plate C 35) which are not contained in Table III, because this region is given in more detail in Tables V and VI :

Hε (4), 4070 *C*III + 4076 *O*II, (2), *Hδ* (7), *Hγ* (7), 4363 [*O*III] (10), 4471 *He*I (0), 4515 *N*III (0), 4601 *N*II (1), 4640 *N*III, etc. (7), 4686 *He*II (3), *Hβ* (7), 4959 [*O*III] (7), 5007 [*O*III] (10), 5755 [*N*II] (7), 5876 *He*II (1), 6300 [*O*I] (0), *Hα* (10).

R, Remarks, in the last column :

C = Lines which seem certainly present.
P = Lines probably present too in the same band.
U = Identification or measured band-detail uncertain.
N = Observed or suspected (*N*?) nebular lines.†

Table III

Nos.	*w*	λ_c	*I*	*O*	λ_L	*M*		I_L	*R*
1–2	18	3125	2	*O*III	3115·7	$3p^3S - 4d^3P^\circ$	1–0	4	*C N*
				*O*III	3121·7	$3p^3S - 4d^3P^\circ$	1–1	5	*C N*
				*O*III	3132·9	$3p^3S - 4d^3P^\circ$	1–2	6	*C N*
				*O*II	3134·8	$e^4D^\circ - w^4P$	$3\frac{1}{2}$–$2\frac{1}{2}$	10	*P*

* The positions of some measured ultra-violet bands may be displaced (or bands may be masked) by the presence of strong absorption bands of atmospheric ozone in this region. (Fowler and Strutt list them in *Proc. Roy. Soc.*, A, **93**, 577, 1917.)

† According to an unpublished recent list for which I am indebted to the kindness of Dr. R. H. Stoy. (Since published in *L.O.B.*, 480.)

Nos.	w	λ_c	I	O	λ_L	M		I_L	R
3–4	18	3162	−1	*Si* IV	3149·6	$4^2P^\circ - 4^2D$	$\frac{1}{2}$–$1\frac{1}{2}$	6	*U*
				N V	3161	$5P - 6S$			*P*
				Si IV	3165·7	$4^2P^\circ - 4^2D$	$1\frac{1}{2}$–	8	*C*
5	...	3228	−1	*Si* III	3230·6	$4p^3P^\circ - 5s^3S$	0–1	3	*U*
				Si III	3234·0	$4p^3P^\circ - 5s^3S$	1–1	5	*U*
				O III	3238·6	$3p^3D - 3d^1P^\circ$	1–1	5	*C*
				Si III	3241·7	$4p^3P^\circ - 5s^3S$	2–1	6	*U*
6–7	16	3261	−1	*O* III	3261·0	$3p^3D - 3d^3F^\circ$	2–3	8	*C*
				O III	3265·5	$3p^3D - 3d^3F^\circ$	3–4	10	*C*
				O III	3267·3	$3p^3D - 3d^3F^\circ$	1–2	5	*C*
8–10	26	3304	0	*O* III	3299·4	$3s^3P^\circ - 3p^3S$	0–1	3	*C N*
				O III	3312·3	$5s^3P^\circ - 3p^3S$	1–1	5	*C N*
11–12	8	3330	0	*O* III	3330·4	$3s^5P - 3p^5P^\circ$	1–2	4	*C*
				O III	3333·0	$3s^5P - 3p^5P^\circ$	2–3	4	*C*
13–14	14	3342	2	*O* III	3340·7	$3s^3P^\circ - 3p^3S$	2–1	6	*C N*
				[*Ne* III]	3342·8	$^1D - ^1S$			*P N*
				[*Ne* V]	3345·8	$3P - ^1D$	1–		*P*
				O IV	3348·1	$3s'^2P - 3p'^2D$	1–2	2	*P*
				O IV	3349·1	$3s'^2P - 3p'^2D$	2–3	3	*P*
				O III	3350·7	$3s^5P - 3p^5P^\circ$	2–1	3	*P*
				O III	3351·0	$3s^5P - 3p^5P^\circ$	3–3	4	*P*
15–16	8	3360	0	*O* III	3362·4	$3s^5P - 3p^5P^\circ$	3–2	4	*P*
17–18	12	3383	1	*O* IV	3381·3	$3p^4P - 3p^4D^\circ$	$1\frac{1}{2}$–$2\frac{1}{2}$	4	*P N*
				O III	3382·7	$3p^5P^\circ - 3d^5D$	2–3	3	*C*
				O III	3383·8	$3p^5P^\circ - 3d^5D$	1–2	2	*C*
				O III	3385·0	$3p^5P^\circ - 3d^5D$	3–4	4	*C*
				O IV	3385·6	$3p^4P - 3p^4D^\circ$	$2\frac{1}{2}$–$3\frac{1}{2}$	6	*P N*
19–20	12	3408	1	*O* IV	3403·6	$3p^2P^\circ - 3d^2D$	$\frac{1}{2}$–$1\frac{1}{2}$	3	*C*
				O IV	3409·8	$3p'^4P - 3d'^4D$	3–3	2	*P*
				O IV	3411·8	$3p^2P^\circ - 3d^2D$	$1\frac{1}{2}$–$2\frac{1}{2}$	4	*C*
				O IV	3413·7	$3p^2P^\circ - 3d^2D$	$1\frac{1}{2}$–$1\frac{1}{2}$	1	*P*
21–22	16	3427	3	[*Ne* V]	3425·8	$^3P - ^1D$	2–		*P N*
				O III	3428·7	$3p^3P - 3d^3P^\circ$	1–2	3	*C N*
				O III	3430·6	$3p^3P - 3d^3P^\circ$	2–1	4	*C N*
23–25	16	3444	2	*N* IV	3443	$3s'^3P - 3p'^3P$	1–2		*U*
				O III	3444·1	$3p^3P - 3d^3P^\circ$	2–2	5	*C N*
26–27	23	3490	1	*N* IV	3478·7	$3s^3S - 3p^3P^\circ$	1–2	7	*C*
				N IV	3483·0	$3s^3S - 3p^3P^\circ$	1–1	5	*C*
				N IV	3484·9	$3s^3S - 3p^3P^\circ$	1–0	3	*C*

Nos.	w	λ_c	I	O	λ_L	M		I_L	R
				Si III	3486·9			6	*P*
				O IV	3489·8	$3s'^2P - 3p'^2D$	2–3	1	*U*
				O IV	3492·2	$3s'^2P - 3p'^2D$	1–2		*U N*
29–30	(32)	3549	0	*O* IV	3560·4	$3p'^2D - 3d'^2F$	2–3	1	*U*
				O IV	3563·4	$3p'^2D - 3d'^2F$	3–4	2	*U*
30–31	(93)	3612	0	*Si* III	3590·5	$4p^1P^\circ - 4d^1D$	1–2	8	*U*
				C III	3609·4	$4p^3P^\circ - 5d^3D$	0, 1, 2–1, 2, 3	5	*U*
31–32	(41)	3678	0	*N* IV	3689	$3p'^3D - 3d'^3P$	2–1		*U*
				O III	3695·4	$3s^5P - 5p^5D^\circ$	1–2	4	*U*
32–34	(33)	3715	1	*O* III	3698·7	$3s^5P - 3p^5D^\circ$	2–3	5	*U*
				O III	3702·8	$3p^3P - 3d^3D^\circ$	0–1	5	*C*
				O III	3703·4	$3s^5P - 3p^5D^\circ$	3–4	5	*P*
				O III	3704·7	$3s^5P - 3p^5D^\circ$	1–1	3	*P*
				O III	3707·2	$3p^3P - 3d^3D^\circ$	1–2	6	*C*
				O III	3712·5	$3s^5P - 3p^5D^\circ$	2–2	2	*P*
				O II	3712·8	$y^4P - e^4S^\circ$	$\frac{1}{2}$–1$\frac{1}{2}$	7	*P*
				N IV	3714	$D - P$	3–2		*U*
				O III	3715·1	$3p^3P - 3d^3D^\circ$	2–3	6	*C*
				O III	3720·9	$3s^5P - 3p^5D^\circ$	3–3	3	*P*
				$H\mu$	3721·9	$2^2P^\circ - 14^2D$			*P N*
				O IV	3725·8	$3p'^4D - 3d'^4F$	2–3	2	*P*
				[*O* II]	3726·2	$^4S - {}^2D$	–1$\frac{1}{2}$		*C N*
				O II	3727·3	$y^4P - e^4S^\circ$	1$\frac{1}{2}$–1$\frac{1}{2}$	8	*P*
				[*O* II]	3728·9	$^4S - {}^2D$	–2$\frac{1}{2}$		*C N*
				O IV	3729·0	$3p'^4D - 3d'^4F$	3–4	3	*P*
34–35	8	3736	1	$H\lambda$	3734·4	$2^2P^\circ - 13^2D$			*P N*
				O IV	3736·8	$3p'^4D - 3d'^4F$	4–5	4	*C*
36–37	(23)	3762	2	*O* II	3749·5	$y^4P - e^4S^\circ$	2$\frac{1}{2}$–1$\frac{1}{2}$	9	*P*
				H_κ	3750·2	$2^2P^\circ - 12^2D$			*P N*
				N III	3754·6	$3s^4P^\circ - 3p^4S$	1$\frac{1}{2}$–1$\frac{1}{2}$	6	*C*
				O III	3754·7	$3s^3P^\circ - 3p^3D$	1–2	7	*C N*
				O III	3757·2	$3s^3P^\circ - 3p^3D$	0–1	5	*C N*
				O III	3759·9	$3s^3P^\circ - 3p^3D$	2–3	9	*C N*
				Si IV	3762·4	$4^2D - 5^2P^\circ$	1$\frac{1}{2}$, 2$\frac{1}{2}$–1$\frac{1}{2}$	4	*P*
				$H\iota$	3770·6	$2^2P^\circ - 11^2D$			*P N*
				N III	3771·1	$3s^4P^\circ - 3p^4S$	2$\frac{1}{2}$–1$\frac{1}{2}$	7	*C*
				Si IV	3773·1	$4^2D - 5^2P^\circ$	1$\frac{1}{2}$–$\frac{1}{2}$	3	*P*
				O III	3774·0	$3s^3P^\circ - 3p^3D$	1–1	6	*C*
38–39	22	3801	0	*O* III	3791·3	$3s^3P^\circ - 3p^3D$	2–2	6	*C*
				Si III	3791·4	$4p^3P^\circ - 4d^3D$	0–1	3	*P*
				Si III	3796·1	$4p^3P^\circ - 4d^3D$	1–	4	*P*
				$H\theta$	3797·9	$2^2P^\circ - 10^2D$			*C N*
				Si III	3806·6	$4p^3P^\circ - 4d^3D$	2–	5	*P*

Nos.	w	λ_c	I	O	λ_L	M		I_L	R
40–41	17	3838	1	$H\eta$	3835·4	$2^2P^\circ - 9^2D$			$C\ N$
				NII	3838·4	$3p^3P - 4s^3P^\circ$	2–2	5	P
					3840				N?
42–43	17	3872	4	[NeIII]	3868·7	$^3P - {}^1D$	2–		$C\ N$
44–45	15	3892	3	$CIII$	3885·9	$4d^3D - 5f^3F^\circ$	2–3	2	P
				HeI	3888·6	$2^3S - 3^3P^\circ$	1–	10	$C\ N$
				$H\zeta$	3889·1	$2^2P^\circ - 8^2D$	...	...	$C\ N$
				$CIII$	3889·1	$4d^3D - 5f^3F^\circ$	3–4	3	P

(b) *The Blue Region from $H\epsilon$ to* λ 5050

Table IV gives the results of the measurement of four selected plates T456, 466, 500, 530.

Table V is arranged in the same way as Table III, and gives the identifications between $H\epsilon$ and λ 5050.

TABLE IV

No.	530	500	466	456	λ_m	Description
46			3966·7		3966·7	*v e*
47			3973·3		3973·3	*r e*
48			3992·5		3992·5	*v e*
49			3996·7		3996·7	*r e*
50			4020·8		4020·8	*v e*
51			4030·4		4030·4	*r e*
52			4038·1		4038·1	*v e*
53			4045·1		4045·1	*r e*
54			4062·2		4062·2	*v c f v ext u*
55	4066·2	4065·3			4065·8	*v e s p u*
56	4076·2	4077·2	4078·9		4077·4	*r e s p*
57	4091·4	4091·9	4089·8	4091·3	4091·1	*v e*
58	4093·2	4095·9			4094·6	*c n em*
59	4096·3	4096·6		4096·5	4096·5	*v e s p, d g u*
60	4102·6	4102·8		4101·5	4102·3	*r e s p*
61	4107·0	4106·8	4108·0		4107·3	*r e*
62			4112·5		4112·5	*v e*
63			4121·0		4121·0	*r e*
64			4136·7		4136·7	*v e*
65			4143·4		4143·4	*r e*
66			4144·5		4144·5	*v e*
67			4154·1		4154·1	*r e*
68			4170·3		4170·3	*v e f v ext u*
69	4181·2	4177·7	4177·4		4178·8	*r e f v ext u*
70			4180·1		4180·1	*v e*

No.	530	500	466	456	λ_m	Description
71		4189·5	4188·3		4188·9	*r e*
72			4190·7		4190·7	*v e r ext*
73		4196·6			4196·6	*n m u*
74	4197·6	4198·3			4198·0	*n m*
75	4199·9	4199·7			4199·8	*n m*
76		4201·6	4200·4		4201·0	*r e r ext*
77	4233·7	4232·2	4230·1		4232·0	*v e s p u*
78	4243·2	4242·1	4242·2		4242·5	*r e s p u*
79		4251·6			4251·6	*n m*
80		4258·4	4255·0		4256·7	*v e*
81	4260·5	4260·8		4260·0	4260·4	*d l*
82	4267·3			4266·9	4267·1	*d g*
83	4272·0	4273·3	4272·0	4270·7	4272·0	*r e u*
84		4277·2			4277·2	*d g u*
85		4285·6			4285·6	*d g u*
86	4310·6	4310·7	4308·6		4310·0	*v e*
87	4321·9	4322·0	4320·6		4321·5	*r e*
88		4328·1	4327·3	4330·3	4328·6	*v e*
89		4330·5			4330·5	*d g*
90	4333·5				4333·5	*d l u*
91	4336·6				4336·6	*d g u*
92	4339·6				4339·6	*d g u*
93	4342·4			4342·0	4342·2	*d g u*
94	4343·9				4343·9	*d g u*
95	4344·9				4344·9	*r e s p*
96		4346·3	4346·2	4346·3	4346·3	*c n p*
97	4349·1	4350·8	4348·8		4349·6	*d g u*
98				4349·9	4349·9	*v e*
99	4354·1				4354·1	*d l*
100	4356·5				4356·5	*v e s p*
101	4361·8			4361·9	4361·8	*d g vy u*
102	4365·0				4365·0	*d g*
103	4368·3			4370·1	4369·2	*r e s p*
104	4372·5		4371·6	4371·3	4371·8	*d l u*
105		4374·6			4374·6	*d l*
106				4378·8	4378·8	*d g u*
107	4384·3	4382·6	4386·9		4384·6	*r e s r ext vy u*
108		4405·0			4405·0	*v e*
109	4409·6	4409·4	4405·0	4407·3	4407·5	*v e s p*
110	4414·0	4414·3		4413·9	4414·1	*r e s p*
111	4419·4	4422·1	4419·2		4420·2	*r e r f ext*
112	4427·3	4426·2	4424·7		4426·0	*v e s p*
113		4434·6	4432·7		4433·6	*r e s p u*
114		4440·8	4440·6	4440·5	4440·6	*g, r e u*
115	4441·8				4441·8	*2 d l u*
116		4444·4			4444·4	*v e s p*
117		4448·0			4448·0	*r e s p*
118	4455·6		4453·5	4451·7	4452·6	*r e*
119	4458·7		4460·1		4459·4	*d g u*
120		4461·1		4462·4	4461·7	*v e*
121	4463·6				4463·6	*d l*

No.	530	500	466	456	λ_m	Description
122	4465·1				4465·1	*v e*
123	4467·3				4467·3	*v e s p u*
124				4467·4	4467·4	*d g u*
125	4471·0			4472·9	4472·0	*r e s p u*
126	4474·1				4474·1	*d g u*
127	4478·0	4477·4	4477·0		4477·5	*r e*
128	4481·9			4481·6	4481·8	*v e f v ext*
129	4489·0			4490·0	4489·5	*r e f v ext*
130	4494·8	4493·8			4494·3	*v e*
131	4501·4	4501·4	4502·6		4501·8	*r e*
132	4506·2	4504·2		4501·3	4503·9	*v e s p*
133		4509·4			4509·4	*d g*
134		4510·4			4510·4	*d g*
135	4513·4				4513·4	*d g*
136	4516·4				4516·4	*d g*
137	4522·8	4522·3	4521·8	4520·5	4521·4	*r e s p*
138				4527·0	4527·0	*v e*
139	4527·4				4527·4	*d l, v e f em*
140		4533·4			4533·4	*r e f em*
141	4545·2	4542·3		4548·9	4544·1	*r e vy u*
142			4545·4		4545·4	*v e vy u*
143			4552·9		4552·9	*r e*
144			4563·5		4563·5	*v e u*
145				4560·8	4560·8	*v e s p u*
146				4569·0	4569·0	*r e s p u*
147			4573·7		4573·7	*r e u*
148	4586·9	4585·9	4584·4	4587·1	4586·1	*v e*
149				4593·1	4593·1	*v e*
150	4594·7	4594·5	4597·2		4595·5	*d l*, 466 : *vy u*
151				4600·9	4600·9	*d g u*
152	4601·3	4600·7			4601·0	*c f n m*
153	4602·6	4603·0			4602·8	*c s n m*
154	4606·2	4505·1			4605·6	*c s n m*
155	4608·4	4607·3			4607·8	*c f n m*
156				4608·2	4608·2	*r e*
157	4610·7		4610·7		4610·7	*c vy f n m*
158	4618·9				4618·9	*n m*
159			4622·9	4622·1	4622·5	*v e*
160	4625·2	4624·3			4624·8	*n m u*
161	4627·0				4627·0	*d l u*
162	4628·0				4628·0	*r e*
163	4633·5				4633·5	*c n m*
164	4637·9				4637·9	*c n m*
165	4644·8				4644·8	*c n m*
166	4649·2	4648·0	4651·3	4648·3	4649·2	*v e r ext*
167	4653·6				4653·6	*v e s p*
168	4658·4				4658·4	*r e s p*
169	4665·8	4661·7	4661·0	4659·4	4662·0	*r e r ext, d l*
170	4668·6		4669·7	4672·0	4670·1	*d g u*
171		4672·7			4672·7	*v e f v ext*
172	4679·4	4677·5			4678·4	*v e*

No.	530	500	466	456	λ_m	Description
173				4684·7	4684·7	*d g u*
174	4692·6	4690·8		4690·5	4691·3	*r e*
175	4699·3	4698·7	4696·0	4697·9	4698·0	*v e s p u*
176		4703·0		4708·8	4705·9	*d g vy u*
177	4715·9	4715·8	4714·7		4715·5	*r e s p u*
178		4722·6			4722·6	*v e f em*
179			4728·4		4728·4	*r e f em*
180			4732·8		4732·8	*v e u*
181			4740·4		4740·4	*r e u*
182			4745·3		4745·3	*v e*
183			4748·5		4748·5	*d g u*
184			4760·7		4760·7	*v e f p*
185			4790·9		4790·9	*r e f p*
186			4795·9		4795·9	*v e u*
187			4804·4		4804·4	*r e*
188			4805·7		4805·7	*v e*
189			4812·5		4812·5	*r e*
190			4838·7		4838·7	*v e u*
191			4840·8		4840·8	*d l*
192		4849·6	4848·1	4848·5	4848·7	*v e*
193				4863·9	4863·9	*d g u*
194		4867·6	4874·9	4869·4	4870·6	*r e*
195			4884·0		4884·0	*d l u*
196			4907·8		4907·8	*v e*
197			4915·6		4915·6	*r e*
198			4937·7		4937·7	*d g*
199		4944·5	4945·4	4946·0	4945·3	*v e*
200	4948·2				4948·2	*d l*
201	4953·6				4953·6	*c n em*
202				4959·3	4959·3	*d l*
203	4960·9				4960·9	*c f n em*
204		4968·8	4972·6	4965·3	4968·9	*r e*
205		4991·2	4991·2	4992·3	4991·6	*v e*
206	4997·2				4997·2	*v e s p*
207				5009·4	5009·4	*d l*
208	5012·1			5011·6	5011·8	*r e s p*
209		5019·1	5022·6	5017·3	5019·7	*r e*
210			5035·3		5035·3	*v e u*
211			5057·7		5057·7	*r e*

TABLE V

Nos.	w	λ_c	I	O	λ_L	M		I_L	R
46–47	(6·6)	3970·0	2	OIII	3961·6	$3s^1P^\circ - 3p^1S$	1–0	8	*P*
				[*Ne*III]	3967·5	$3P - {}^1D$	1–		*P N*
				*He*II	3968·5	$4^2F^\circ - 14^2G$			*P*
				$H\epsilon$	3970·1	$2^2P^\circ - 7^2D$			*C N*

Nos.	w	λ_c	I	O	λ_L	M		I_L	R
48–49	(4·2)	3994·6	0	*N*II	3995·0	$3s^1P^\circ - 3p^1P$	1–1	10	*C*
				*O*IV	3995·2	$3p'^4P - 3d'^4P$	3–3	1	*U*
50–51	(9·6)	4025·6	0	*He*II	4025·6	$4^2F^\circ - 13^2G$			*P N*
				*He*I	4026·2	$2^3P^\circ - 5^3D$	2, 1–3, 2, 1	5	*P N*
				*He*I	4026·4	$2^3P^\circ - 5^3D$	0–1	1	*P N*
52–53	(7·0)	4041·6	0	*N*II	4035·1			4	*P*
				*N*II	4041·3			5	*C*
				*N*II	4043·5			3	*P*
55–56	(10·0)	4071·2	4	*C*III	4067·9	$4f\,^3F^\circ - 5g^3G$	2–3	6	*C*
				[*S*II]	4068·6	$^4S - ^2P$	$-1\frac{1}{2}$		*U N*
				*C*III	4068·9	$4f\,^3F^\circ - 5g^3G$	3–4	7	*C*
				*O*II	4069·9	$e^4D^\circ - z^4F$	$1\frac{1}{2}$–$2\frac{1}{2}$	6	*P*
				*C*III	4070·4	$4f\,^3F^\circ - 5g^3G$	4–5	8	*C*
				*O*II	4072·2	$e^4D^\circ - z^4F$	$2\frac{1}{2}$–$3\frac{1}{2}$	8	*C*
				*O*II	4075·9	$e^4D^\circ - z^4F$	$3\frac{1}{2}$–$4\frac{1}{2}$	10	*C*
				[*S*II]	4076·2	$^4S - ^2P$	$-\frac{1}{2}$		*U N*
57–61	(15·6)	4099·2	8	*Si*IV	4088·9	$4^2S - 4^2P^\circ$	$\frac{1}{2}$–$1\frac{1}{2}$	10	*P*
				*N*III	4097·3	$3s^2S - 3p^2P^\circ$	$\frac{1}{2}$–$1\frac{1}{2}$	10	*C N*
				*He*II	4100·1	$4^2F^\circ - 12^2G$			*P*
				Hδ	4101·8	$2^2P^\circ - 6^2D$			*C N*
				*N*III	4103·4	$3s^2S - 3p^2P^\circ$	$\frac{1}{2}$–$\frac{1}{2}$	9	*C N*
62–63	(8·5)	4116·8	1	*Si*IV	4116·1	$4^2S - 4^2P^\circ$	$\frac{1}{2}$–$\frac{1}{2}$	8	*C*
				*O*V	4120	$3s'^3P - 3p'^3D$	1–2		*U*
				*C*III	4122·0	$4^1P^\circ - 5^1D$	1–2	3	*P*
				*O*V	4123·9	$3s'^3P - 3p'^3D$	2–3	2	*P*
64–65	(6·7)	4140·0	2	*He*I	4143·8	$2^1P^\circ - 6^1D$	1–2	2	*U N*
66–67	(9·6)	4149·3	2	*C*III	4152·4	$p.p^3D - 5^3F^\circ$	1–2	2	*P*
				*S*II	4153·0	$4p^4D^\circ - 4d^4F$	$2\frac{1}{2}$–$3\frac{1}{2}$	10	*U*
				*O*II	4153·3	$e^4P^\circ - x^4P$	$1\frac{1}{2}$–$2\frac{1}{2}$	7	*C*
				*C*III	4156·5	$p.p^3D - 5^3F^\circ$	2–3	3	*P N?*
				*O*V	4158·8	$3p'^3S - 3d'^3P$	1–2		*P*
				*S*II	4162·6	$4p^4D^\circ - 4d^4F$	$3\frac{1}{2}$–$4\frac{1}{2}$	10	*U*
				*C*III	4162·8	$p.p^3D - 5^3F^\circ$	3–4	4	*P N?*
68–69	8·5	4174·6	2	*S*II	4174·0			4	*C*
				*N*IV	4174	$3p'^3S - 3d'^3P$	1–1		*U*
				*S*II	4174·2			5	*C*
70–71	8·8	4184·5	3	*N*IV	4183	$3p'^3S - 3d'^3P$	1–2		*U*
				*O*II	4185·5	$e^2F^\circ - z^2G$	$2\frac{1}{2}$–$3\frac{1}{2}$	8	*P*
				*C*III	4187·0	$4f\,^1F^\circ - 5g^1G$	3–4	10	*C N?*
				*O*II	4189·8	$e^2F^\circ - z^2G$	$3\frac{1}{2}$–$4\frac{1}{2}$	10	*P*

Nos.	w	λ_e	I	O	λ_L	M		I_L	R
72–76	10·3	4195·9	3	*N*III	4195·7	$4p^2P^\circ - 5s^2S?$	$\frac{1}{2}$–$\frac{1}{2}$	5	*C*
				*He*II	4199·9	$4^2F^\circ - 11^2G$			*P N*
				*N*III	4200·0	$4p^2P^\circ - 5s^2S?$	$1\frac{1}{2} - \frac{1}{2}$	6	*C*
77–78	10·5	4237·2	1	*N*II	4237·0			6	*C*
				*N*II	4241·8			8	*C*
79	...	4251·6	0	*C*III	4247·6	$3p^1D - 5p^1P^\circ$	2–1	1	*U*
80–83	15·3	4264·3	5	*O*IV	4262	$3p'^2D - 3d'^2F$	3–4		*U*
				*S*II	4266·9			4	*U*
				*C*II	4267·0	$3d^2D - 4f^2F^\circ$	$1\frac{1}{2}$–$2\frac{1}{2}$	8	*C N*
				*C*II	4267·3	$3d^2D - 4f^2F^\circ$	$2\frac{1}{2}$–$2\frac{1}{2}$, $3\frac{1}{2}$	10	*C N*
				*S*II	4267·3			4	*U*
				*S*II	4267·8	$4p^4P^\circ - 4d^4D$	$1\frac{1}{2}$–$2\frac{1}{2}$	6	*U*
86–87	11·5	4315·8	2	*O*II	4317·2	$y^4P - e^4P^\circ$	$\frac{1}{2}$–$1\frac{1}{2}$	8	*C*
				*O*II	4319·6	$y^4P - e^4P^\circ$	$1\frac{1}{2}$–$2\frac{1}{2}$	8	*C*
87–88	7·1	4325·1	0	*C*III	4325·7	$p.s^1P^\circ - p.p^1P$	1–1	7	*C*
				*N*III	4328·2	$3p^4D - 3d^4D^\circ$	$1\frac{1}{2}$–$1\frac{1}{2}$	3	*P*
88–97	21·0	4339·1	10	*N*V	4335	$6S - 7P$			*U*
				*N*III	4335·5	$3p^4D - 3d^4D^\circ$	$2\frac{1}{2}$–$2\frac{1}{2}$	4	*P*
				*O*II	4336·9	$y^4P - e^4P^\circ$	$1\frac{1}{2}$–$1\frac{1}{2}$	6	*P*
				*He*II	4338·7	$4^2F^\circ - 10^2G$			*P N*
				*N*III	4339·5	$3p^4D - 3d^4D^\circ$	$2\frac{1}{2}$–$1\frac{1}{2}$	3	*P*
				Hγ	4340·5	$2^2P^\circ - 5^2D$			*C N*
				*O*II	4345·6	$y^4P - e^4P^\circ$	$1\frac{1}{2}$–$\frac{1}{2}$	7	*P*
				*O*II	4347·4	$y^2D - f^2D^\circ$	$1\frac{1}{2}$–$1\frac{1}{2}$	5	*P*
				*N*III	4348·4	$3p^4D - 3d^4D^\circ$	$3\frac{1}{2}$–$3\frac{1}{2}$	5	*C*
				*O*II	4349·4	$y^4P - e^4P^\circ$	$2\frac{1}{2}$–$2\frac{1}{2}$	8	*P N?*
				*O*II	4351·3	$y^2D - f^2D^\circ$	$2\frac{1}{2}$–$2\frac{1}{2}$	6	*P N?*
98–104	21·9	4360·8	15	*C*III	4361·8	$4f'^1G - 5g'^1H?$		2	*U*
				[*O*III]	4363·2	$2p^1D - 2p^1S$	2–0		*C N*
				*O*II	4366·9	$y^4P - e^4P^\circ$	$2\frac{1}{2}$–$1\frac{1}{2}$	7	*P*
				*C*III	4368·1	$4f'^3G - 5g'^3H?$		4	*P*
104–107	12·8	4378·2	2	*C*II	4374·3			2	*U*
				*N*III	4379·1	$4f^2F^\circ - 5g^2G$		10	*C N?*
				*O*III	4379·6			3	*P*
108–111	15·2	4412·6	3	*C*II	4411·2			2	*U*
				*C*II	4411·5			2	*U*
				*O*II	4414·9	$z^2P - e^2D^\circ$	$1\frac{1}{2}$–$2\frac{1}{2}$	10	*C N*
				*O*II	4417·0	$z^2P - e^2D^\circ$	$\frac{1}{2}$ – $1\frac{1}{2}$	8	*C N*
112–114	14·6	4433·3	1	*N*II	4432·7	$3d^3P^\circ - 4f^3D$	2–3	6	*C*

Nos.	w	λ_c	I	O	λ_L	M		I_L	R
115–118	10·8	4447·2	2	*C* IV	4440·0	$5P - 6D$	1–2		*P*
				C IV	4441·2	$5P - 6D$	2–3		*P*
				O II	4443·0	$e^2F^\circ - y^2F$	$2\frac{1}{2}$–$2\frac{1}{2}$	5	*P*
				N II	4447·0	$3p^1D - 3d^1P^\circ$	2–1	10	*C*
				O II	4448·2	$e^2F^\circ - y^2F$	$3\frac{1}{2}$–$3\frac{1}{2}$	6	*P*
				O II	4452·4	$z^2P - e^2D^\circ$	$1\frac{1}{2}$–$1\frac{1}{2}$	6	*P*
122–127	12·4	4471·3	4	*He* I	4471·5	$2^3P^\circ - 4^3D$	2, 1–3, 2, 1	6	*C N*
				He I	4471·7	$2^3P^\circ - 4^3D$	0–1	1	*P N*
128–129	7·7	4485·6	0	*N* IV	4479	$3s'^3P - 3p'^3S$	0–1		*U*
130–131	7·5	4498·0	1	*N* IV	4495	$3s'^3P - 3p'^3S$	1–1		*U*
132–137	17·5	4512·7	4	*N* III	4510·9	$3s^4P^\circ - 3p^4D$	$\frac{1}{2}$,$1\frac{1}{2}$–$1\frac{1}{2}$,$2\frac{1}{2}$	6	*C*
				N III	4514·9	$3s^4P^\circ - 3p^4D$	$2\frac{1}{2}$–$3\frac{1}{2}$	7	*C*
				C III	4516·5	$4p^3P - 5s^3S$		4	*P*
138–141	17·1	4535·6	2	*N* IV	4528	$3s'^3P - 3p'^3S$	2–1		*U*
				N II	4530·4	$3p^1D - 3d^1F^\circ$	2–3	5	*C*
				He II	4541·6	$4^2F^\circ - 9^2G$			*C N*
142–143	(7·5)	4556·7	1	*N* II	4552·5	$3p^1D - 3d^3F^\circ$	2–3	4	*C*
				Si III	4552·6	$4s^3S - 4p^3P^\circ$	1–2	9	*C*
				O V	4554·3	$3p'^1P - 3d'^1D$			*P*
144–147	10·2	4568·6	1	*Si* III	4567·8	$4s^3S - 4p^3P^\circ$	1–1	7	*C*
					4571·5				*N*
				Si III	4574·8	$4s^3S - 4p^3P^\circ$	1–0	4	*C*
148–149	7·0	4589·6	1	*O* II	4591·0	$y^2D - e^2F^\circ$	$2\frac{1}{2}$–$3\frac{1}{2}$	9	*C*
				O II	4596·2	$y^2D - e^2F^\circ$	$1\frac{1}{2}$–$2\frac{1}{2}$	8	*C*
149–156	15·1	4600·7	4	*N* II	4601·5	$3s^3P^\circ - 3p^3P$	1–2	8	*C N?*
				N V	4604·5	$3s^3S - 3p^2P$	$\frac{1}{2}$–$1\frac{1}{2}$		*C*
157	...	4610·7	2	*N* II	4607·2	$3s^3P^\circ - 3p^3P$	0–1	7	*C N?*
				O II	4609·4			4	*P*
				O II	4610·1			3	*P*
				N II	4613·9	$3s^3P^\circ - 3p^3P$	1–1	6	*C*
158	...	4618·9	2	*C* II	4618·8			2	*U*
				N V	4620·5	$3s^2S - 3p^2P^\circ$	$\frac{1}{2}$–$\frac{1}{2}$		*C*
159–162	5·5	4625·2	3	*N* II	4621·4	$3s^3P^\circ - 3p^3P$	1–0	7	*U*
162–166	21·2	4638·6	9	*N* II	4630·6	$3s^3P^\circ - 3p^3P$	2–2	10	*C*
				Si IV	4631·4	$5^2F^\circ - 6^2G$	$2\frac{1}{2}$, $3\frac{1}{2}$–$3\frac{1}{2}$, $4\frac{1}{2}$	3	*P*
				N III	4634·2	$3p^2P^\circ - 3d^2D$	$\frac{1}{2}$–$1\frac{1}{2}$	8	*C N*
				O II	4638·9	$y^4P - e^4D^\circ$	$\frac{1}{2}$–$1\frac{1}{2}$	6	*P*

Nos.	w	λ_c	I	O	λ_L	M		I_L	R
				*N*III	4640·6	$3p^2P^\circ - 3d^2D$	$1\frac{1}{2}$–$2\frac{1}{2}$	10	*C N*
				*O*II	4641·8	$y^4P - e^4D^\circ$	$1\frac{1}{2}$–$2\frac{1}{2}$	9	*C*
				*N*III	4641·9	$3p^2P^\circ - 3d^2D$	$1\frac{1}{2}$–$1\frac{1}{2}$	3	*P*
				*N*II	4643·1	$3s^3P^\circ - 3p^3P$	2–1	8	*C*
				*C*IV	4646·5	$5D - 6F$			*P*
				*C*III	4647·4	$3^3S - 3^3P^\circ$	1–2	10	*C N*
				*C*IV	4648·7	$5d^2D - 6f^2F^\circ$	$1\frac{1}{2}$, $2\frac{1}{2}$–$2\frac{1}{2}$, $3\frac{1}{2}$		*P*
				*O*II	4649·1	$y^4D - e^4D^\circ$	$2\frac{1}{2}$–$3\frac{1}{2}$	10	*P*
166–169	(12·8)	4655·6	4	*C*III	4650·2	$3^3S - 3^3P^\circ$	1–1	9	*C N*
				*O*II	4650·8	$y^4P - e^4D^\circ$	$\frac{1}{2}$–$\frac{1}{2}$	6	*P*
				*C*III	4651·4	$3^3S - 3^3P^\circ$	1–0	8	*C*
				*Si*IV	4654·1	$5^2G - 6^2H^\circ$	$3\frac{1}{2}$, $4\frac{1}{2}$–$4\frac{1}{2}$, $5\frac{1}{2}$	4	*P*
				*C*IV	4656·5	$5F - 6G$			*P*
				*C*IV	4658·6	$5G - 6H$			*C N*
				*O*II	4661·6	$y^4P - e^4D^\circ$	$1\frac{1}{2}$–$1\frac{1}{2}$	9	*P*
				*C*III	4663·5	$3s^3P^\circ - 3p^3P$	1–0	3	*P*
				*C*IV	4664·5	$5F - 6D$			*U*
				*C*III	4665·9	$3s^3P^\circ - 3p^3P$	2–2	5	*U*
171–172	5·7	4675·6	1	*C*III	4673·9	$3s^3P^\circ - 3p^3P$	2–1	4	*C*
				*O*II	4676·2	$y^4P - e^4D^\circ$	$2\frac{1}{2}$–$2\frac{1}{2}$	8	*C*
172–174	12·9	4684·7	6	*C*IV	4685·4	$6G, H - 8H, I$			*P*
				*He*II	4685·8	$3^2D - 4^2F^\circ$			*C N*
175–177	17·5	4706·8	2	*O*II	4699·2	$f^2D^\circ - y^2F$	$2\frac{1}{2}$–$3\frac{1}{2}$	7	*C*
				*O*II	4705·4	$e^2D^\circ - z^2F$	$2\frac{1}{2}$–$3\frac{1}{2}$	8	*C N*?
				*O*II	4710·0	$e^2D^\circ - z^2F$	$1\frac{1}{2}$–$2\frac{1}{2}$	5	*C*
				[*A*IV]	4711·4	$^4S - ^2D$	$-2\frac{1}{2}$		*P N*
				*He*I	4713·1	$2^3P^\circ - 4^3S$	2, 1–1	3	*P N*
				*He*I	4713·4	$2^3P^\circ - 4^3S$	0–1	1	*P N*
178–179	5·8	4725·5	0	[*Ne*IV]	4725·5	$^2D - ^2P$	$1\frac{1}{2}$–$\frac{1}{2}$		*C N*
180–181	7·6	4736·8	0	*N*IV	4733	$3p'^3D - 3d'^3D$	2–2		*U*
				[*A*V]	4738	$^1D - ^1S$			*U*
				[*A*IV]	4740·2	$^4S - ^2D$	$-1\frac{1}{2}$		*C N*
184–185	(30·2)	4775·8	0	*N*II	4779·7	$3p^3D - 3d^3D^\circ$	1–1	4	*P*
				*O*IV	4783·4	$3p^4P - 3d^4D^\circ$	$1\frac{1}{2}$–$\frac{1}{2}$	4	*P*
				*C*IV	4785·6	$5D - 6P$			*U*
				*N*II	4788·1	$3p^3D - 3d^3D^\circ$	2–2	5	*P*
186–187	8·5	4800·2	0	*O*IV	4798·2	$3p^4P - 3d^4D^\circ$	$2\frac{1}{2}$–$1\frac{1}{2}$	5	*P*
				*N*II	4803·3	$3p^3D - 3d^3D^\circ$	3–3	6	*C*
188–189	6·8	4809·1	1	*N*II	4810·3	$3p^3D - 3d^3D^\circ$	3–2	2	*U*

Nos.	w	λ_c	I	O	λ_L	M		I_L	R
192–194	21·9	4859·7	8	*N*III	4858·7	$3p^4D - 3d^4F^\circ$	$\frac{1}{2}$–$1\frac{1}{2}$	2	*P*
				*N*III	4858·9	$3p^4D - 3d^4F^\circ$	$1\frac{1}{2}$–$2\frac{1}{2}$	3	*P*
				*He*II	4859·4	$4^2F^\circ - 8^2G$			*P*
				*N*III	4861·3	$3p^4D - 3d^4F^\circ$	$2\frac{1}{2}$–$3\frac{1}{2}$	4	*P*
				Hβ	4861·3	$2^2P^\circ - 4^2D$			*C N*
				*N*III	4867·2	$3p^4D - 3d^4F^\circ$	$3\frac{1}{2}$–$4\frac{1}{2}$	5	*P*
199–204	23·6	4957·1	10	*N*V	4943	$6F - 7G$			*U*
				*N*V	4945	$6G, H - 7H, I$			*U*
				*N*V	4952	$6F - 7D$			*U*
				[*O*III]	4958·9	$2p^3P - 2p^1D$	1–2		*C N*
205–209	28·1	5005·6	12	*N*II	4994·4	$3p^3S - 3d^3P^\circ$	1–1	6	*P*
				*N*II	5001·1	$3p^3D - 3d^3F^\circ$	1–2	7	*P*
				*N*II	5001·5	$3p^3D - 3d^3F^\circ$	2–3	8	*P*
				*N*II	5005·1	$3p^3D - 3d^3F^\circ$	3–4	10	*C*
				[*O*III]	5006·8	$2p^3P - 2p^1D$	2–2		*C N*
				*N*II	5007·3	$3p^3S - 3d^3P^\circ$	1–2	7	*P*
				*N*II	5010·6	$3s^3P^\circ - 3p^3S$	1–1	6	*P*
				*He*I	5015·7	$2^1S - 3^1P^\circ$	0–1	6	*C N*
				*C*IV	5015·9	$5P - 6S$	1–		*P*
				*C*IV	5017·7	$P - S$	2–		*P*
210–211	22·4	5046·5	1	*N*II	5045·1	$3s^3P^\circ - 3p^3S$	2–1	8	*C*

Some of the identifications of Table V, together with those of Tables III and VI, may be summarised as follows :—

H.—The Balmer lines are, on the ultra-violet plates, certainly traceable from *Hα* down to *Hη* ; *Hθ* is blended with *O*III 3791 ; *Hι* and *Hκ* may contribute to the broad band Nos. 36–37. And *Hλ*, *Hμ*, etc. may be already too faint to be contained in the bands Nos. 34–35, 32–34, 31–32.

*He*I seems to be represented mostly only for intensities 5 or more. λλ 4471 and 5876 are strong and unblended, while 3889 blends with *Hζ*, 4026 with *He*II 4025·6, 5015 with [*O*III] 5007, etc.

*He*II.—With the exception of the strong λ 4686, which may be blended with a *C*IV-line too, all other *He*II-lines in the blue and red region are contained in blends, and in the weak ultra-violet plates not traceable at all.

*C*II.—The multiplet $3d^2D$–$4f^2F^\circ$ is very strong ; the presence of the other *C*II-lines of intensity 2 or more is indicated.

*C*III.—Strongly present are 4067–70, 4187, and especially 4647–51, 3^3S–3^3P°, but also most of the other *C*III-lines contribute to blends.

*C*IV.—The strong pair 5801–5812 constitutes, perhaps blended with *N*IV, the bright band centred at λ 5806 ; other *C*IV-lines in blends present too.

NII.—Some bands are exclusively due to *N*II, like Nos. 48–49, 52–53, 77–78, 238–239; other strong *N*II-lines are common in blends.

NIII.—All stronger lines represented in blends.

NIV is mainly responsible for the band Nos. 26–27, but perhaps blended with *Si*III; also *N*IV 5812 may blend with *C*IV-lines.

NV.—Strong maxima of this atom appear at λλ 4604 and 4620; other *N*V-lines perhaps blending with [*O*III] 4959.

[*NII*].—λ 5755 is one of the strongest lines; the two other forbidden *N*II-lines in our range are found in nebulæ at λ 6548·1 and λ 6583·6, but they do not seem to be present in this nova stage in view of the fact that *Hα* covers only the range λ 6545–82.

OII causes exclusively bands like Nos. 86–87, 148–149; other strong *O*II-lines take part in blends.

OIII characterises especially the ultra-violet spectrum; and seems to be, except perhaps the multiplet $3p^5D^\circ - 3d^5F$, nearly completely represented.

OIV.—The bands Nos. 19–20 and 34–35 are explained by *O*IV (λλ 3404, 3412, 3736), and others are present in blends.

OV.—Many *O*V-lines may appear blended.

[*OI*].—λλ 5577, 6300, 6363, visible since the first weeks after the appearance of the nova, are, 6300 being the brightest, still present.

[*OII*].—λλ 3726·2 and 3728·9, which were observed in nebulæ too, seem to be contained in the red extension of the *O*III-band λ 3702–15.

[*OIII*] is furnishing with λλ 4363, 4959, 5007 three of the brightest bands of the whole spectrum.

[*FIII*].—This identification of λ 5721·9 is doubtful. See Bowen (*loc. cit.*), Stoy (*loc. cit.*), Spencer Jones (*M.N.*, **94**, 821, 1934).

NeIII.—All three lines, Bowen, *loc. cit.*, seem to be present; but only λ 3868·7 is unblended, while λ 3342 blends with *O*III, and λ 3968 with *Hε*.

[*NeIV*].—λ 4725·5 seems present; the only predicted [*Ne*IV]-line, which may be observed in nebulæ; doubtful, Stoy (*loc. cit.*).

[*NeV*].—λ 3345·8 and λ 3425·8, both observed in nebulæ, may perhaps be present in blends.

AlII.—This identification of the band Nos. 246–247 is doubtful (see C. H. Payne, *Z. f. Aph.*, **7**, 14, 1933); perhaps the unidentified nebular line at 6234 is alone responsible for it, as well as *N*II 6442.

SiIII.—The strong multiplet $4s^3S - 4p^3P^\circ$ is present. The other strong *Si*III-lines are apparently represented in blends.

SiIV.—Its lines, also the strongest at λ 4088·8 and λ 4116·1, are all blended.

SII.—The band Nos. 68–69 can be explained by *S*II, whilst other *S*II-lines may belong to blends only.

[*SII*].—The presence of these nebular lines λλ 4068, 4076, is somewhat doubtful.

[*AIV*].—λ 4740·2, probably blended with [AV], seems to account for the band centred at λ 4736·8. The other predicted [*A*IV]-line, which may be observed at λ 4711·4 in nebulæ too, although this is not quite certainly established, may be present blended with *O*II and *He*I.

[*AV*] explains the band at λ 6434, also observed in nebulæ; the other line λ 4738 blends with [*A*IV].

[*CaV*].—This identification of Nos. 238–239 seems perhaps somewhat doubtful. But Bowen, *loc. cit.*, has given as predicted value λ 6085·9; Stoy, *loc. cit.*, has observed the line in N.G.C. 7027; and Spencer Jones the same in Nova Pictoris, see *M.N.*, **94**, 821, 1934, and *Cape Ann.*, *loc. cit.* A second [*Ca*V]-line predicted at 5308·9 may be identical with an unexplained line at 5305·0 in Wolf-Rayet stars (see Payne, *Z. f. Aph.*, **7**, 14, 1933), or with λ 5313 in the nebulæ N.G.C. 7027 and 7662, but this is in the difficult green part of the nova plates. And the third predicted line λ 3996·3 may well blend with *N*II 3994·6.

(c) *The Red Region from* λ 5400 *to Hα*

Between λ 5050 and λ 5400 is a gap, in which no lines have been found on any of the Cambridge plates later than 1935 April. The two plates T466 and T467 have especially well-exposed red parts, and were therefore used for the Table VI, which gives again the identifications: *

Table VI.

Nos.	*w*	λ_c	*I*	*O*	λ_L	*M*		I_L	*R*
212–213	25	5416	2	*He*II	5411·6	$4^2F^\circ - 7^2G$			*C N*
				*O*V	5418·3	$3p^3P - 3d^3P^\circ$	1–2		*P*
214–215	23	5466	1	*O*IV	5453	$4p'^2P - 4d'^2D$	2–3		*U*
				*N*II	5462·6	$3p^3P - 3d^3P^\circ$	1–1	3	*U*
				*O*V	5473	$3p'^3P - 3d'^3P$	2–2		*C*
216–217	19	5492	0	*N*II	5480·1	$3p^3P - 3d^3P^\circ$	2–1	3	*P*
				*N*II	5495·7	$3p^3P - 3d^3P^\circ$	2–2	5	*C*
218–219	20	5534	1	*N*II	5530·3	$3s^5P - 3p^5D^\circ$	1–2	4	*C*
				*N*II	5535·4	$3s^5P - 3p^5D^\circ$	3–4, 1–1	5	*C*
220–221	17	5574	2	*O*V	5573	$3p^3P - 3d^3D$	0–1		*P*
				[*O*I]	5577·4	$2p^1D - 2p^1S$	2–0		*C N*
222–223	15	5591	0	*O*V	5585·3	$3p^3P^\circ - 3d^3D$	1–1		*P*
				*O*III	5592·4	$3s^1P^\circ - 3p^1D$	1–2	6	*C*
224–225	(12)	5620	0		5621				*U N?*
226–227	7	5664	1	*C*II	5662·5	$3s^4P^\circ - 3p^4S$	$2\frac{1}{2}$–$1\frac{1}{2}$	4	
				*N*II	5666·6	$3s^3P^\circ - 3p^3D$	1–2	8	*N?*

* Fig. 2 contains a reproduction of the red part of T467; the continuous spectrum shown in this plate is due to the use of a lamp for following the star image on the slit plate.

Nos.	w	λ_c	I	O	λ_L	M		I_L	R
228–229	14	5679	5	*N*II	5676·0	$3s^3P^\circ - 3p^3D$	0–1	6	*P*
				*N*II	5679·6	$3s^3P^\circ - 3p^3D$	2–3	10	*C N?*
				*N*II	5686·2	$3s^3P^\circ - 3p^3D$	1–1	6	*P*
229–230	(12)	5692	0	*C*III	5695·8	$3p^1P^\circ - 3d^1D$	1–2	5	*C N*
230–231	(42)	5718	2	*N*II	5710·8	$3p^1P^\circ - 3p^3D$	2–2	6	*C*
				[*F*III]	5721·9	$^2D - ^2P$	$2\frac{1}{2}$–		*U N*
				[*F*III]	5732·0	$^2D - ^2P$	$1\frac{1}{2}$–		*U*
				*N*IV	5734	$3p'^1P - 3d'^1D$			*U*
					5737				*N?*
				*Si*III	5739·8	$4s^1S - 4p^1P^\circ$	0–1	8	*U*
231–232	32	5754	8	[*N*II]	5754·8	$2p^1D - 2p^1S$	2–0		*C N*
233–234	21	5806	3	*C*IV	5801·5	$3S - 3P$	–2		*C N*
				*N*IV	5812	$3p'^3P - 3d'^3P$	1–2		*C N*
				*C*IV	5812·1	$S - P$	–1		*C*
235–236	20	5876	7	*He*I	5875·6	$2^3P^\circ - 3^3D$	2, 1–3, 2, 1	10	*C N*
				*He*I	5876·0	$2^3P^\circ - 3^3D$	0–1	1	*P N*
237–238	(10)	5926	0	*N*II	5927·8	$3p^3P - 3d^3D^\circ$	0–1	4	*C*
				*N*II	5931·8	$3p^3P - 3d^3D^\circ$	1–2	7	*C*
238–239	19	5940	1	*N*II	5941·7	$3p^3P - 3d^3D^\circ$	2–3	8	*C*
240–241	22	6088	1	[*Ca*V]	6085·9	$^3P - ^1D$	1–		*U N*
242–243	(45)	6169	0	*N*II	6167·8	$3d^3F^\circ - 4p^3D$	4–3	4	*C*
				*N*II	6173·4	$3d^3F^\circ - 4p^3D$	3–2	3	*C*
244–245	24	6204	0		6205				*N?*
246–247	(33)	6238	0	*Al*II	6231·8	$4^3P^\circ - 4^3D$	1–2	7	*U*
					6234				*N*
				*N*II	6242·5	$3d^1F^\circ - 4p^1D$	3–2	5	*C*
				*Al*II	6243·6	$4^3P^\circ - 4^3D$	2–3	10	*U*
					6246				*N?*
248–249	19	6299	6	[*O*I]	6300·2	$2p^3P - 2p^1D$	2–2		*C N*
250–251	27	6361	3	[*O*I]	6363·9	$2p^3P - 2p^1D$	1–2		*C N*
251–252	(13)	6380	0	*N*II	6379·6	$3s^3P^\circ - 3p^1D$	1–2	5	*P*
253–254	(39)	6434	0	[*A*V]	6435	$^3P - ^1D$	1–		*U N*
254–255	(52)	6479	0	*N*II	6482·1	$3s^1P^\circ - 3p^1D$	1–2	8	*C*
256–257	37	6560	10	*He*II	6560·2	$4^2F^\circ - 6^2G$			*P*
				Hα	6562·8	$2^2P^\circ - 3^2D$			*C N*

Summary

1. The nebular spectrum of Nova Herculis between 1935 May and December has been analysed.

2. The origin of about a hundred bright and mostly blended bands in the region λ 3120–λ 6580 can be attributed to the following 30 atoms :—

H, *He*I, *He*II, *C*II, *C*III, *C*IV, *N*II, *N*III, *N*IV, *N*V, [*N*II], *O*II, *O*III, *O*IV, *O*V?, [*O*I], [*O*II]?, [*O*III], [*F*III]?, [*Ne*III], [*Ne*IV]?, [*Ne*V]?, *Al*III?, *Si*III, *Si*IV, *S*II, [*S*II]?, [*A*IV], [*A*V], [*Ca*V]?.

3. The structure of the main bright bands has been studied in detail.

In conclusion the writer wishes to thank Professor F. J. M. Stratton sincerely for his kind helpfulness throughout this investigation.

ON A SUSPECTED VARIATION IN THE LIGHT OF β CANIS MINORIS.

W. M. Smart, M.A., D.Sc.

(Communicated by the Director, Cambridge Observatory)

1. *Summary.*

(i) The Harvard photo-electric observations of the difference in magnitude between Procyon and β Canis Minoris are first corrected for atmospherical extinction.

(ii) The resulting light-curve is of the Cepheid type, with a period of $0^d.09$; the difference between maximum and minimum is $0^m.02$.

(iii) It is shown that the observational data are in good accordance with the period-luminosity law, and it is concluded that β Canis Minoris is a short-period Cepheid or cluster-type variable.

2. In a recent publication * Dr. W. A. Calder has given the magnitude differences between Procyon and β Canis Minoris measured with the photo-electric photometer mounted on the Wyeth reflector at the Oak Ridge station of Harvard Observatory. A rotating sector was used to diminish the light of Procyon. Altogether 39 observations were made during an interval of about five hours (approximately half on each side of the meridian) on the night of 1935 January 18–19 with the object of testing a possible short-period light-variation in Procyon as suggested from spectroscopic observations. The stars are fairly close together in the sky, and the atmospherical extinction correction should, in consequence, be comparatively small throughout the period of observation. This consideration led Calder to ignore the correction, although he noticed that the observations showed

* *Harvard Bulletin*, **899**, 13, 1935.

a fairly continuous increase in the magnitude difference for the two stars, amounting to about $0^m.03$ between the beginning and the end of the set of measures. This increase, he stated, can hardly be attributed to extinction, and he concluded, as a result of the very small magnitude differences recorded, that there was no evidence of a short-period variation in the light of either star. But, having regard to the simultaneous zenith distances of the stars, the magnitude change just mentioned ($0^m.03$) is just of the order of magnitude to be expected for atmospherical extinction during the period of observation, and large enough to conceal small fluctuations in brightness, if any. As the observations appear to be extremely accurate, I considered them worthy of a more detailed analysis, and the result was the detection of a clearly defined periodicity in the light of either Procyon or of β Canis Minoris.

3. Let m_1 and m_2 denote the magnitudes of Procyon and of β Canis Minoris as observed, and let $\Delta m_0 = m_2 - m_1$. The values of Δm_0 are given in Calder's paper. Let Δm denote the difference in magnitude when the correction for atmospherical extinction has been applied. Then we have

$$\Delta m = \Delta m_0 + fF, \qquad (1)$$

where

$$F = \sec z_1 - \sec z_2,$$

z_1 and z_2 being the zenith distances and f the extinction factor. Table I contains the values of the zenith distances of the two stars, together with the values of F and of fF, at intervals of 30^m in the local sidereal time.* I have assumed that $f = 0^m.40$, corresponding to a night of very good seeing, and, further, that f is constant throughout the period of observation. The spectral types of the stars are respectively F5 and B8, and it is possible that the values of f are slightly different for the two types, although with the Cambridge photometer I have failed to detect any appreciable variation.

TABLE I

Local Sidereal Time	Zenith Distance		F	fF
	Procyon	β Can. Min.		
h m	° ′	° ′		m
5 6	49 51	45 38	0·121	+0·048
5 36	45 44	41 36	0·095	38
6 6	42 15	38 16	0·077	31
6 36	39 33	35 48	0·064	26
7 6	37 50	34 26	0·054	22
7 36	37 15	34 20	0·045	18
8 6	37 50	35 28	0·038	15
8 36	39 33	37 40	0·033	13
9 6	42 15	40 56	0·027	11
9 36	45 44	44 51	0·022	9
10 6	49 51	49 18	0·017	7
10 36	54 26	54 10	0·011	4

* I am indebted to Mr. H. E. Green for the calculations of the zenith distances. The position of the Oak Ridge station was assumed to be 42° 39′ N., 4^h 46^m W., as deduced from a statement in *Harvard Report*, 1931, that the station is 26 miles north-west of Harvard Observatory.

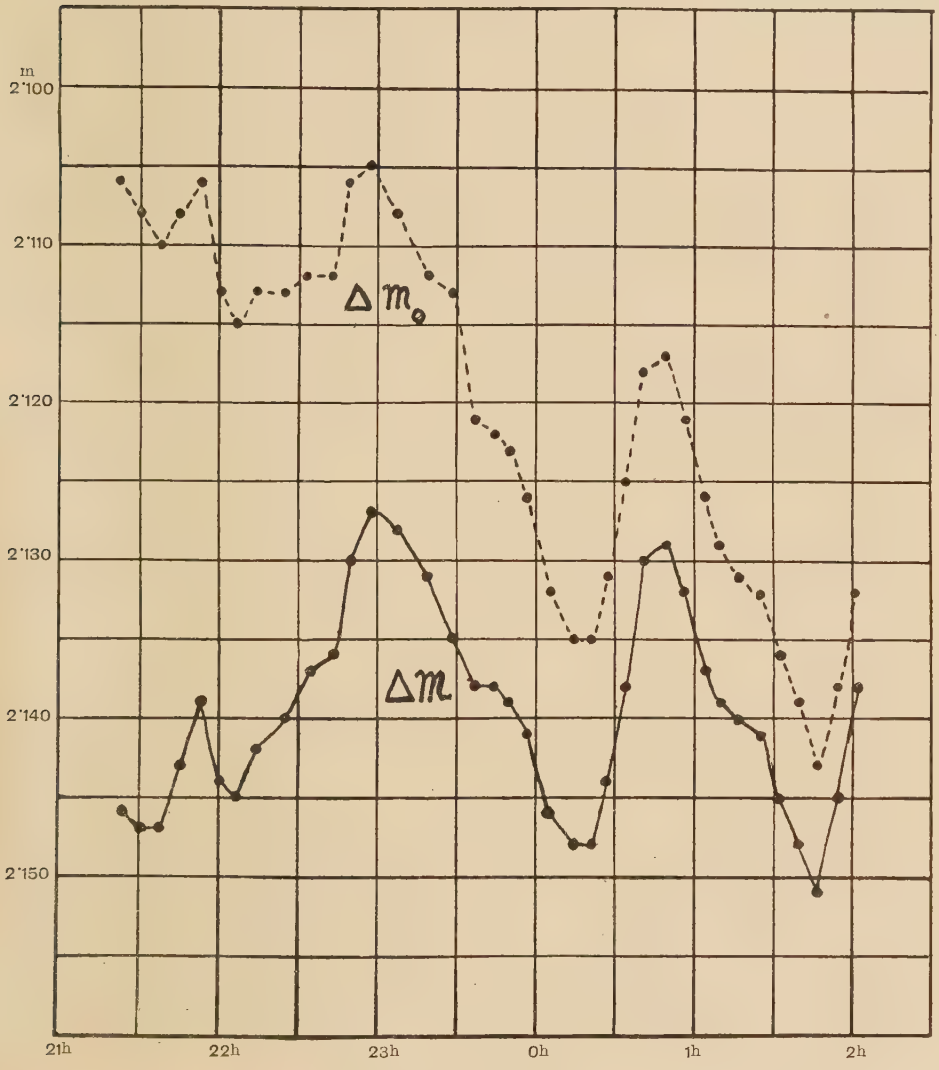
TABLE II

Local Sidereal Time	Δm_0 (H.B. 899) $2^m.000 +$	Extinction Correction fF	Δm (Corrected) $2^m.100 +$	Δm (Mean of Three Consecutive Observations) $2^m.100 +$	E.S.T.	Δm_0 (Mean of Three Consecutive Observations) $2^m.100 +$
h m		m			h m	
5 20	0.108	+0.043	0.051			
29	094	40	34	0.046	21 24	0.006
34	115	39	54	47	31	8
41	116	37	53	47	39	10
53	100	34	34	43	46	8
5 57	108	33	41	39	21 54	6
6 4	111	31	42	44	22 1	13
13	119	30	49	45	7	15
17	116	29	45	42	15	13
27	104	27	31	40	25	13
43	118	25	43	37	34	12
47	113	24	37	36	43	12
6 55	104	23	27	30	50	6
7 3	103	22	25	27	22 59	5
14	108	20	28	28	23 8	8
23	113	19	32	31	19	12
34	115	18	33	35	29	13
44	122	17	39	38	37	21
47	125	16	41	38	44	22
7 56	118	16	34	39	50	23
8 1	126	15	41	41	23 57	26
10	135	14	49	46	0 6	32
22	135	14	49	48	15	35
27	134	13	47	48	23	35
33	135	13	48	44	28	31
40	124	12	36	38	35	25
46	117	12	29	30	42	18
8 55	113	12	25	29	50	17
9 3	122	11	33	32	0 57	21
8	127	11	38	37	1 4	26
15	130	10	40	39	11	29
25	130	10	40	40	18	31
30	132	9	41	41	26	32
37	133	9	42	45	33	36
47	143	8	51	48	41	39
53	142	8	50	51	48	43
9 58	144	7	51	45	1 55	38
10 8	127	7	34	38	2 2	32
15	124	6	30			

In the second column of Table II, Calder's values of Δm_0 are given; the third column contains the values of the extinction correction, fF, and the fourth column contains the corrected magnitude difference, Δm, as calculated from (1). In the fifth column are the smoothed values of Δm, obtained by taking the mean of three consecutive values, and in the next

column are the corresponding Eastern Standard Times. The last column contains the smoothed values of Δm_0 as for Δm.

The values of Δm, as given in the fifth column of Table II, are shown



The abscissæ are E.S.T.

plotted on the accompanying diagram, and the curve based on the uncorrected values, Δm_0, of the last column is given for comparison.

4. The Δm curve shows, I think unmistakably, that one of the two stars is a short-period variable, the period being about 1·8 hours and the range in magnitude about $0^m.02$. The curve bears a strong resemblance to a typical Cepheid variation, with a comparatively slow decrease in brightness followed by a rapid increase. The only discordant points are the first four—that is, those between $21^h.4$ and $21^h.8$ E.S.T.—and it is precisely at this part

of the curve where discordances might reasonably be expected; here the zenith distances of the stars are large—between 40° and 48°—and, what is more important, the differences in the zenith distances of the two stars are greatest, leading to the largest values of the extinction correction and to the greatest uncertainty if f is not known accurately. At the other end of the curve the zenith distances are large, but they are very nearly equal for the two stars, with the consequence that the extinction correction is comparatively small. Between 23^h and 2^h the precise value of the factor f is relatively unimportant, and the character of the curve would remain practically unaltered for any value of f between $0^m{\cdot}35$ and $0^m{\cdot}45$. The two maxima and the two minima are remarkably accordant, and the agreement could be made almost perfect by taking f to be $0^m{\cdot}44$.

If we regard Procyon as photometrically invariable, the curve represents a light variation of β Canis Minoris, which is definitely of the Cepheid type. If β Canis Minoris is invariable, the light-curve should be drawn inverted, in which case we have a slow increase in brightness followed by a rapid decline. I believe that the first interpretation is the correct one. Moreover, the observational facts are in good accordance with the period-luminosity law for Cepheids and cluster variables. The parallax of β Canis Minoris, according to Schlesinger's *Bright Star Catalogue*, is $0''{\cdot}024$, and with the visual magnitude $3^m{\cdot}1$, the absolute visual magnitude is exactly $0^m{\cdot}0$. Applying the appropriate bolometric correction,* we find the absolute bolometric magnitude to be $-0^m{\cdot}7$. Also, the effective temperature, T_e, of a B8 star is 13,500° (*loc. cit.*, p. 141). The period-luminosity relation is given by Jeans † as

$$\log P + 0{\cdot}23m + 3 \log T_e = \text{constant}, \qquad (2)$$

where P is the period in days and m the absolute bolometric magnitude. Taking the constant to be 11·35—the mean of the values in Table 32 (*loc. cit.*)—the value of the period is found to $0^d{\cdot}13$. Considering that the above calculation is in the nature of an extrapolation from a formula based on periods exceeding $0^d{\cdot}3$, the agreement with the observed period of $0^d{\cdot}08$ may be accounted satisfactory. This conclusion is also arrived at if Shapley's visual period-luminosity curve ‡ is examined; the curve flattens out in the neighbourhood of absolute visual magnitude $0^m{\cdot}0$ and becomes almost parallel to the log P axis.

Observatory, Cambridge:
1935 *January* 4.

* Eddington, *Internal Constitution of the Stars*, 138 (Table 15), 1926.
† *Astronomy and Cosmogony*, p. 378, 1928.
‡ *Star Clusters*, 129, 1930.

THE STABILITY OF A PARTICLE IN A GRAVITATIONAL FIELD.

V. V. Narlikar.

1. *Introduction*

In this paper some general criteria are obtained for the stability of a position of equilibrium of a test-particle in a gravitational field. When applied to a spherical universe, every point of which is shown to be a possible position of equilibrium of a particle, the criteria give that in the expanding universe, only, all possible positions are stable. This may possibly explain why the universe has chosen to expand rather than contract.

A particle is here considered to be in a position of stable equilibrium when, if slightly disturbed, it never leaves the neighbourhood of its initial position. Several illustrations of stability and instability are considered. Incidentally by a particle we mean only a mass-particle.

2. *Criteria of Stability*

Consider a line-element of the form

$$ds^2 = -g_1 dx_1^2 - g_2 dx_2^2 - g_3 dx_3^2 + g_4 dx_4^2, \tag{1}$$

where g_1, g_2, g_3, g_4 are positive functions of the variables. It will be assumed that the geodesics give the motion of test-particles in the field of (1). Taking x_4 as the independent variable and denoting a differentiation with regard to x_4 by a dot, three equations for a geodesic may be obtained, one of which runs as follows :—

$$\begin{aligned}\ddot{x}_1 &+ \dot{x}_1[2\{14, 1\} - \{44, 4\}] + \{44, 1\} \\ &- \dot{x}_1[\{11, 4\}\dot{x}_1^2 + \{22, 4\}\dot{x}_2^2 + \{33, 4\}\dot{x}_3^2] \\ &+ \dot{x}_1^2[\{11, 1\} - 2\{14, 4\}] + 2\dot{x}_1\dot{x}_2[\{12, 1\} - \{24, 4\}] + 2\dot{x}_1\dot{x}_3[\{13, 1\} - \{34, 4\}] = 0. \end{aligned} \tag{2}$$

The curvilinear brackets in the above have the usual meaning. At a position of equilibrium,

$$\dot{x}_1 = \dot{x}_2 = \dot{x}_3 = 0, \qquad \ddot{x}_1 = \ddot{x}_2 = \ddot{x}_3 = 0, \tag{3}$$

so that

$$\{44, 1\} = \{44, 2\} = \{44, 3\} = 0. \tag{4}$$

The conditions (4) follow from (2) and the other equations of a geodesic. Evaluating the brackets, we have

$$\frac{\partial g_4}{\partial x_1} = \frac{\partial g_4}{\partial x_2} = \frac{\partial g_4}{\partial x_3} = 0 \tag{5}$$

at a point of equilibrium. The points of equilibrium are therefore given by the condition that g_4 is stationary there. When g_4 does not involve the space co-ordinates the conditions (5) are fulfilled everywhere and every point of the field becomes a possible position of equilibrium. It is for

this reason that in Einstein's or Lemaître's universe equilibrium becomes possible at every point. It appears from the Newtonian analogue that in the external field of a distribution of matter there are no positions of equilibrium. Conditions (5) are not therefore expected to be fulfilled at any point in the external field of a distribution. In the internal field, however, there may be positions where g_4 is stationary.

Let $(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ be a position of equilibrium so that when a slight disturbance (ξ_1, ξ_2, ξ_3) is given at any instant the geodesics give

$$\ddot{\xi}_1 + \lambda_1 \dot{\xi}_1 + A_{11}\xi_1 + A_{12}\xi_2 + A_{13}\xi_3 = 0, \tag{6}$$

$$\ddot{\xi}_2 + \lambda_2 \dot{\xi}_2 + A_{21}\xi_1 + A_{22}\xi_2 + A_{23}\xi_3 = 0, \tag{7}$$

$$\ddot{\xi}_3 + \lambda_3 \dot{\xi}_3 + A_{31}\xi_1 + A_{32}\xi_2 + A_{33}\xi_3 = 0, \tag{8}$$

where

$$\lambda_1 = 2\{14, 1\} - \{44, 4\}, \qquad \lambda_2 = 2\{24, 2\} - \{44, 4\},$$

$$\lambda_3 = 2\{34, 3\} - \{44, 4\}, \qquad A_{11} = \frac{1}{2g_1}\frac{\partial^2 g_4}{\partial x_1^2}, \qquad A_{12} = \frac{1}{2g_1}\frac{\partial^2 g_4}{\partial x_1 \partial x_2},$$

$$A_{21} = \frac{1}{2g_2}\frac{\partial^2 g_4}{\partial x_1 \partial x_2}, \quad \text{etc., etc. all defined at } \bar{x}_1, \bar{x}_2, \bar{x}_3.$$

During the time that λ_r and A_{rs} are not appreciably altered the equations (6), (7) and (8) will always keep ξ_1, ξ_2, ξ_3 small provided *

$$\begin{aligned}(m^2 + \lambda_1 m + A_{11})(m^2 + \lambda_2 m + A_{22})(m^2 + \lambda_3 m + A_{33}) - A_{12}A_{21}(m^2 + \lambda_3 m + A_{33}) \\ - A_{13}A_{31}(m^2 + \lambda_2 m + A_{22}) - A_{23}A_{32}(m^2 + \lambda_1 m + A_{11}) + A_{12}A_{23}A_{31} + A_{13}A_{32}A_{21} = 0\end{aligned} \tag{9}$$

has no root with a positive real part. We are interested here in the particular case where $A_{rs} = 0$, $r \neq s$. If this is satisfied (9) will not have a root with a positive real part provided

$$\lambda_r \geqslant 0, \qquad A_{rr} \geqslant 0, \qquad \lambda_r + A_{rr} > 0 \quad (r = 1, 2, 3). \tag{10}$$

Conditions (10) are fulfilled if

$$\lambda_1 = \lambda_2 = \lambda_3 = 0 \tag{11}$$

and

$$A_{11}, \qquad A_{22}, \qquad A_{33} > 0. \tag{12}$$

When the field is static (11) is at once satisfied. (12) shows that in a static field the positions of stable equilibrium are those where g_4 is a minimum.

If, however,

$$A_{11} = A_{22} = A_{33} = 0, \tag{13}$$

then by (10), at the points of stable equilibrium,

$$\lambda_1, \qquad \lambda_2, \qquad \lambda_3 > 0 \tag{14}$$

or

$$\frac{\partial}{\partial x_4} \log g_r > \tfrac{1}{2}\frac{\partial}{\partial x_4} \log g_4, \quad (r = 1, 2, 3). \tag{15}$$

* Refer to Routh's *Stability of Motion* (1877), p. 32, for the analytical conditions of stability for a sextic. There may be a pitfall in applying conditions (10) as shown by Routh on p. 96.

In a non-static spherical universe $g_4 = 1$, so that (13) is satisfied and (15) is also satisfied provided R is an increasing function of time.

When $A_{11} = A_{22} = A_{33} = 0$ and also $\lambda_1 = \lambda_2 = \lambda_3 = 0$, as in the Einstein universe, the investigation of stability has to start from the three general equations, one of which is (2).

3. *Applications*

We will now make a few applications of the results reached in the last section. Consider the line-element

$$ds^2 = -\frac{dx_1^2 + dx_2^2 + dx_3^2}{(1 + r^2/4R^2)^2} + (A - B\cos\chi)^2 dt^2, \tag{16}$$

with $r/2R = \tan \chi/2$.

Case (i). $\dot{R} = 0$, $A = 0$, $B = 1$. This gives de Sitter's universe. g_4 is stationary at the origin, and, in particular, as g_4 is a maximum here the position is one of instability. Since the origin of the co-ordinates may be any arbitrary point of de Sitter's universe one finds that although equilibrium is possible at every point, nowhere is it stable. The equations (6), (7), (8) for small motion at $r = 0$ all take the form

$$\ddot{\xi} - \xi/R^2 = 0. \tag{17}$$

Thus, as is well known, a repulsive force makes the equilibrium unstable at every point in the de Sitter universe.

Case (ii). $\dot{R} = 0$, $A = 1$, $B = 0$. This gives the Einstein universe. It has already been remarked that the criteria obtained in the last section fail in this case. For a small motion about $(\bar{x}, y, \bar{z})$ we have three equations each of the form

$$\ddot{\xi}_1 = \frac{\bar{x}}{2R^2} \frac{\dot{\xi}_1^2}{(1 + \bar{r}^2/4R^2)}, \tag{18}$$

where $\bar{r}^2 = \bar{x}^2 + \bar{y}^2 + \bar{z}^2$. By (18) $\dot{\xi}_1$ and $\ddot{\xi}_1$ can both be positive. $\bar{x}$, $\bar{y}$, $\bar{z}$ cannot, therefore, be a position of stable equilibrium.

Case (iii). Consider the line-element

$$ds^2 = -\frac{e^{g(t)}}{(1 + r^2/4R_0^2)^2}[dx_1^2 + dx_2^2 + dx_3^2] + dt^2, \tag{19}$$

where R_0 is a constant, $r^2 = x_1^2 + x_2^2 + x_3^2$ and g is some function of time. Equilibrium is obviously possible at all points. The equations of small motion are of the form

$$\ddot{\xi} + \dot{g}\dot{\xi} = 0. \tag{20}$$

The condition of stability demands that $\dot{g} > 0$. It is obvious therefore that unless the radius of the spherical universe (19) increases, equilibrium cannot be stable at each point.

If $R_0 \to \infty$ and in the limit $g(t)$ takes the form $2kt$, where k is a positive constant, the line-element (19) reduces to de Sitter's universe. If we take the following metric for de Sitter's universe,

$$ds^2 = -e^{2kt}[dx_1^2 + dx_2^2 + dx_3^2] + dt^2, \tag{21}$$

the equations of small motion, which are of the form (20), give that the equilibrium of every position in de Sitter's universe is stable. Thus if the static form of de Sitter's universe is everywhere unstable, the non-static form is everywhere stable. There is no contradiction here. Our criteria of stability are not independent of the space-time frame chosen. What is motion under a force of repulsion leading to instability may also be represented as a general space expansion so that an observation of the particle in equilibrium reveals no instability.

Case (iv). The internal solution for a fluid sphere has been given by Schwarzschild in the form

$$ds^2 = -\frac{dr^2}{(1-r^2/R^2)} - r^2(d\theta^2 + \sin^2\theta d\phi^2) + [\tfrac{3}{2}\cos\chi_0 - \tfrac{1}{2}\cos\chi]^2 dt^2, \quad (22)$$

with $\bar{R}^2 = 8\pi\rho/3$, $R\sin\chi = r$ and $R\sin\chi_0 = r_0$ as the radius. From the form of g_4 it is clear that the only position of equilibrium is the centre of the sphere, where g_4 is minimum and hence the equilibrium is stable. There is only one equation for small motions, viz.

$$\ddot{r} + \frac{3\cos\chi_0 - 1}{4R^2} r = 0. \quad (23)$$

Since the pressure has to be finite everywhere Schwarzschild gives the condition $3\cos\chi_0 > 1$. The equation (23) therefore gives small oscillations about the centre. If χ_0^2 is neglected and the value of R^2 substituted in (23) we get the Newtonian equation

$$\ddot{r} + \frac{4\pi\rho r}{3} = 0$$

for small oscillations about the centre of a homogeneous sphere.

Benares Hindu University, India.

MICROMETRICAL MEASURES OF DOUBLE STARS.

(27TH SERIES.)

W. Milburn.

The micrometrical measures are given as in previous lists, all being made with the 24-inch (Calver) reflector.

Name	B.D.	1900 R.A.	1900 Decl.	P.A.	Dist.	Mags.	Nights	Date 1935 +
	°	h m	° ′	°	″			
Es 1935	+36·5155	0 2·2	+37 7	139·8	8·86	...	2	·921
Es 151	+39·12	3·7	40 5	197·3	6·39	...	2	·751
Es 152	+39·27	8·1	39 40	108·2	7·28	...	2	·827
Es 1936*	+38·22	11·5	38 37	77·2	5·40	...	2	·871

Name	B.D.	1900 R.A.	1900 Decl.	P.A.	Dist.	Mags.	Nights	Date 1935 +	
	°	h m	° ′	°	″				
Es 1602	+40·60	0 15·6	40 17	87·8	7·31	...	2	·942	
Es 222	+38·46	20·5	38 19	158·0	6·23	...	2	·946	
M 1	...	21·5	40 18	235·5	6·14	...	3	·759	
M 2	...	24·3	39 37	232·0	3·39	...	2	·854	
Es 1941	+37·92	29·4	37 44	348·4	2·62	...	2	·972	
Es 1948	+37·288	1 22·8	37 50	246·5	4·88	...	2	·968	
Es 1494	+40·323	29·5	40 52	11·2	5·31	...	1	·975	
Es 166	+39·844	3 35·0	39 23	9·1	4·53	...	2	·961	
M 16	+38·802	39·0	38 50	317·1	8·58	...	2	·976	
M 17	+39·903	53·1	39 52	30·6	8·58	...	2	·961	
Es 1616	+40·911	4 7·5	40 47	234·1	6·14	...	2	·961	
Es 2086	...	14·0	38 22	208·3	3·15	...	1	·966	
Es 2149	+38·911	27·9	38 11	221·6	5·68	...	1	·966	
Es 240	+39·1083	44·0	39 18	217·7	4·47	...	2	·962	
Es 2034	+38·3440	19 4·0	38 41	75·2	4·63	9·5, 13·0	2	·782	
Σ 2481	+38·3466	7·7	38 37	215·2	4·47	7·8, 7·8	2	·665	
M 882	...	20 16·4	39 2	283·6	4·26	10·5, 12·0	1	·862	
Es 2050	+38·4020	16·5	39 2	97·1	6·05	9·5, 12·0	2	·843	AB
...	...	...	...	187·7	11·76	...	2	·843	AC
M 23	+40·4284	38·5	40 19	46·2	7·47	...	3	·830	
M 24	...	39·2	40 14	178·6	5·12	...	2	·814	
M 25	+40·4298	41·1	41 9	210·6	5·06	...	3	·775	
Σ 2731	+39·4331	45·3	39 25	86·9	4·35	...	2	·671	AB
...	...	...	...	175·3	...	...	1	·673	AC
...	...	...	...	247·3	...	...	1	·673	AD
Es 1688	+40·4466	21 12·5	41 4	147·9	4·82	...	3	·776	
Es 260	+38·4525	28·5	38 35	282·1	4·75	9·2, 12·0	2	·757	
M 899	...	29·1	39 11	51·1	3·24	...	2	·671	
Σ 2894	+37·4537	22 14·6	37 16	193·9	15·99	...	3	·684	
Es 2070	...	18·9	36 12	157·2	3·81	...	3	·924	
Es 392	+38·4808	31·4	39 7	224·6	9·49	...	3	·792	Dd
Es 1701	+39·4961	49·4	39 28	66·5	7·69	9·5, 9·6	2	·671	
M 33	+38·4894	49·9	38 15	299·4	6·11	...	3	·924	
Σ 2979	+39·5008	23 3·1	39 15	223·6	2·90	8·5, 10·5	2	·689	
Es 219	+35·5000	13·8	35 48	310·4	5·27	...	3	·924	
Es 2001	+36·3050	16·2	36 36	297·1	6·45	...	2	·773	
Es 1598	...	30·8	40 25	275·0	2·87	...	3	·737	
Es 268	+39·5161	41·8	39 59	269·1	4·48	...	2	·960	
Es 269	+40·5153	44·0	40 46	216·8	7·28	...	2	·783	AB
...	...	...	...	139·5	15·56	...	2	·783	AC
M 35	...	44·6	38 24	223·4	3·95	...	2	·871	

*Note on star marked **

Es 1936 : $0^h\ 11^m{\cdot}5$; misidentified by Espin as +38°·21 (B.D.).

Missing B.D. star

B.D. +39°·4071 : $20^h\ 6^m\ 2^s{\cdot}9$; +39° 52′·6 : mag. 9·4 ; 1935 Oct. 16, no star seen.

New Double Stars

Milburn No.	B.D.	1900 R.A.	1900 Decl.	P.A.	Dist.	Mags.	Nights	Date 1935+	
	°	h m	° ′	°	″				
970*	...	0 34·9	+39 31	347·3	4·01	10·5, 10·6	2	·773	
971	...	52·4	39 14	22·7	4·23	10·0, 12·5	2	·965	
972	...	52·5	39 12	288·9	5·37	11·5, 12·0	2	·965	
973	...	1 46·1	38 47	204·0	3·18	12·5, 13·0	2	·965	
974	...	5 4·3	37 2	294·2	6·82	10·0, 11·5	2	·962	
975	+37·1602	6 45·5	37 11	35·6	2·22	9·5, 9·6	1	·966	
976	...	18 48·4	40 19	289·6	4·29	11·7, 12·3	2	·758	
977	...	19 23·5	39 17	296·0	2·82	11·0, 11·0	3	·761	
978*	...	26·6	39 57	140·2	4·44	10·5, 11·0	2	·782	
979	...	43·5	39 11	206·1	2·16	10·0, 10·2	3	·700	
980	...	43·9	39 11	25·4	3·60	9·8, 10·5	3	·696	
981	...	54·1	38 50	56·3	3·67	10·3, 11·2	2	·801	
982	+39·4100	20 11·5	39 11	185±	20±	9·3, 10·0	1	·767	AB
...	...	...	...	357·7	4·19	C=11·0	2	·769	BC
983	...	46·0	38 38	193·3	3·21	10·5, 12·0	2	·794	
984	...	21 23·5	39 28	95·1	3·12	11·7, 12·7	2	·738	
985	...	23·6	39 18	256·0	3·08	10·5, 11·0	2	·760	
986	...	24·9	39 15	62·9	2·99	12·0, 12·2	4	·763	
987	...	25·9	39 2	258·6	7·22	10·0, 10·2	2	·877	
988	...	28·4	38 36	111·3	1·88	11·0, 12·3	2	·757	
989	...	28·8	38 34	137·0	2·13	10·0, 10·0	2	·794	
990	...	29·2	39 34	235·1	4·04	10·5, 12·0	3	·894	
991	...	48·4	38 13	24·1	5·06	10·0, 10·2	3	·716	
992	...	57·3	38 25	64·1	5·99	11·5, 13·5	3	·851	
993	+39·4791	22 10·9	39 40	189·1	5·99	9·5, 13·0	3	·797	
994	...	14·7	38 42	254·7	3·33	11·1, 13·1	1	·923	
995	...	39·2	39 44	297·7	4·29	10·0, 10·1	2	·760	
996	...	42·4	38 31	173·1	2·39	11·0, 11·3	3	·955	
997	...	23 30·6	40 26	217·5	2·59	10·5, 12·0	4	·740	

Notes on stars marked *

970. AC ; 52°·7 ; 39″ ± ; C=11·0 ; 1935·775.
978. AC ; 310°·4 ; 39″ ± ; C=9·6 ; 1935·782.

The Observatory,
Tow Law, Co. Durham.

ABSORPTION COEFFICIENTS AND HYDROGEN LINE INTENSITIES.*

D. H. Menzel and C. L. Pekeris.

Correction

The statement on p. 102 that our formula (2.26) gives nearly equal opacities for wave-lengths 4000 A. and 6000 A. is true on the wave-length scale. On the frequency scale the two opacities differ by a factor of about two.

* *M.N.*, **96**, 77, 1935.